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HEAT TRANSFER IN THE TURBULENT INCOMPRESSIBLE

BOUNDARY LAYER

III - ARBITRARY WALL TEMPERATURE AND HEAT FLUX

By W. C. Reynolds, W. M. Kays, and S. J. Kline
Stanford University

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON

December 1958

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SUMMARY

Superposition techniques are used to calculate the rate of heat transfer from a flat plate to a turbulent incompressible boundary layer for several cases of variable surface temperature. The predictions of a number of these calculations are compared with experimental heat-transfer rates, and good agreement is obtained. A simple computing procedure for determining the heat-transfer rates from surfaces with arbitrary wall-temperature distributions is presented and illustrated by two examples. The inverse problem of determining the temperature distribution from an arbitrarily prescribed heat flux is also treated, both experimentally and analytically.

INTRODUCTION

This report is the third in a series of four covering a three-year investigation of heat transfer to the turbulent incompressible boundary layer with arbitrary wall-temperature variation (ref. 1).[1] In the first report the experimental apparatus is described, and the results of experiments with constant wall temperature are given (ref. 3). The second report presents the results of experiments and analyses for a step wall-temperature distribution (ref. 4). In the present report the step-function analysis is used as a basis for predicting heat-transfer rates for several cases of variable wall temperature, and the predictions are compared with experimental data. A simple method for handling arbitrary-wall-temperature problems analytically is also presented. The fourth report presents an analysis of the effect of transition point on heat transfer in the turbulent boundary layer and compares the results with experiments (ref. 5).

[1] This nonisothermal heat-transfer work is summarized briefly in reference 2.

Because of the linearity of the boundary-layer energy equation, the heat-transfer characteristics for a step wall-temperature distribution may be superimposed to determine heat-transfer rates for arbitrary-wall-temperature situations. This fact has been pointed out by Klein and Tribus (ref. 6) and by Rubesin (ref. 7). The superposition results in integrals that must be evaluated at each point where the heat-transfer rate is desired. The integrations are easily performed if the wall-temperature variation is relatively simple, such as linear, parabolic, and so forth; but numerical methods are required for more complex cases. One approximate method for treating variable-wall-temperature problems is to express the wall-temperature distribution by a finite number of steps. However, this results in infinite heat-transfer rates at the steps and thus does not yield meaningful results in the region near the discontinuities. A better method, short of exact integration, is to approximate the temperature distribution by a finite number of linear segments or "ramps." Since this approximation is continuous, the heat-transfer rates are everywhere finite, and good results can be obtained with relatively few ramp segments. This technique is described in the present report, and the results compare favorably with experiments.

The inverse problem, wherein the heat-transfer rates are specified and the temperature distribution is to be determined, can also be handled by superposition of steps or ramps. This problem is also discussed in the present report.

This investigation was carried out at Stanford University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

SYMBOLS

$A(a/x)$	function for prescribed temperature problem, $(10/9)B_r(8/9,10/9) - [1 - (a/x)]$
a	location of ramp, ft
b	height of step, ^{o}F or Btu/(hr)(sq ft)
c_p	specific heat at constant pressure, Btu/(lb)(^{o}F)
$D(a/x)$	function for prescribed heat-flux problem, $\left(1 - \frac{a}{x}\right) - \frac{B_r(1/9,20/9) - (a/x)B_r(1/9,10/9)}{\Gamma(1/9)\Gamma(8/9)}$
G	free-stream mass velocity, ρu_∞, lb/(hr)(sq ft)

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SYMBOLS

$A(a/x)$ function for prescribed temperature problem,
$$(10/9)B_r(8/9,10/9) - [1 - (a/x)]$$

a location of ramp, ft

b height of step, °F or Btu/(hr)(sq ft)

c_p specific heat at constant pressure, Btu/(lb)(°F)

$D(a/x)$ function for prescribed heat-flux problem,
$$\left(1 - \frac{a}{x}\right) - \frac{B_r(1/9,20/9) - (a/x)B_r(1/9,10/9)}{\Gamma(1/9)\Gamma(8/9)}$$

G free-stream mass velocity, ρu_∞, lb/(hr)(sq ft)

$g(\xi,x)$	kernel function for prescribed heat flux
$H(l/x)$	function for prescribed heat-flux problem, $1 - \dfrac{B_r(1/9,10/9)}{\Gamma(1/9)\Gamma(8/9)}$
h	local heat-transfer coefficient, $q''/\Delta t$, Btu/(hr)(sq ft)(°F)
$h(\xi,x)$	local heat-transfer coefficient at x due to a step temperature at ξ
k	thermal conductivity, Btu/(hr)(ft)(°F)
l	location of step, ft
m	slope of ramp, °F/ft or Btu/(hr)(cu ft)
Pr	Prandtl number, $\mu c_p/k$
q''	heat flux, Btu/(hr)(sq ft)
q^*	effective q'', Btu/(hr)(sq ft)
Re_x	Reynolds number based on x, Gx/μ
r	$1 - (a/x)^{9/10}$ or $1 - (l/x)^{9/10}$
$S(l/x)$	function for prescribed temperature problem, $\left[1 - (l/x)^{9/10}\right]^{-1/9} - 1$
St	local Stanton number, h/Gc_p
St_H	local Stanton number for constant heat input
St_T	local Stanton number for isothermal plate
T_w	absolute plate temperature, °R
T_∞	absolute free-stream temperature, °R
t	temperature, °F
Δt	$t_w - t_\infty$, °F
Δt^*	effective Δt, °F
t_m	mean temperature of heated strip, °F

Δt_m $t_m - t_\infty$, $^\circ F$

t_w plate temperature, $^\circ F$

t_∞ free-stream temperature, $^\circ F$

u_∞ free-stream velocity, ft/sec

$w(x)$ load distribution

x distance from leading edge, ft

y distance from plate, ft

z variable of integration

$B(a,b)$ beta function, $\int_0^1 z^{a-1}(1 - z)^{b-1}\,dz$

$B_r(a,b)$ beta function, $\int_0^r z^{a-1}(1 - z)^{b-1}\,dz$

$\Gamma(a)$ gamma function, $\int_0^\infty e^{-z}z^{a-1}\,dz = (a-1)!$

μ viscosity, lb/(hr)(ft)

ξ variable of integration

ρ density, lb/cu ft

φ beam deflection, in.

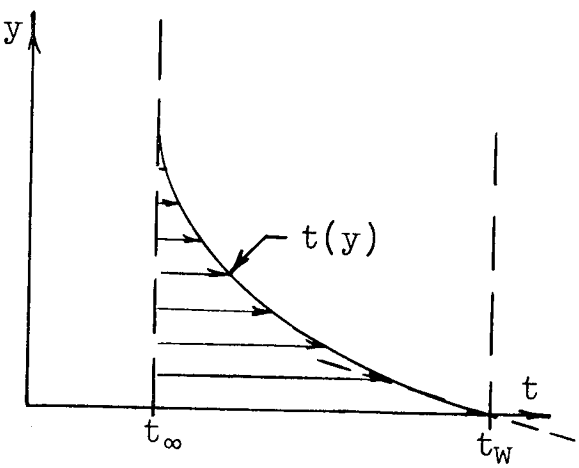
ω beam influence coefficient, in./lb

QUALITATIVE EFFECTS OF VARIABLE WALL TEMPERATURE

Until quite recently, practically all investigations of boundary-layer heat transfer treated the special case of constant surface temperature. Although the assumption of constant temperature greatly simplifies analysis, many important systems involve heat transfer from nonisothermal surfaces, and failure to consider the effects of the nonisothermality can often lead to serious errors in calculated heat-transfer rates. Recent efforts have provided methods for treating the nonisothermal problem, and the calculation of heat-transfer rates is now essentially mathematical.

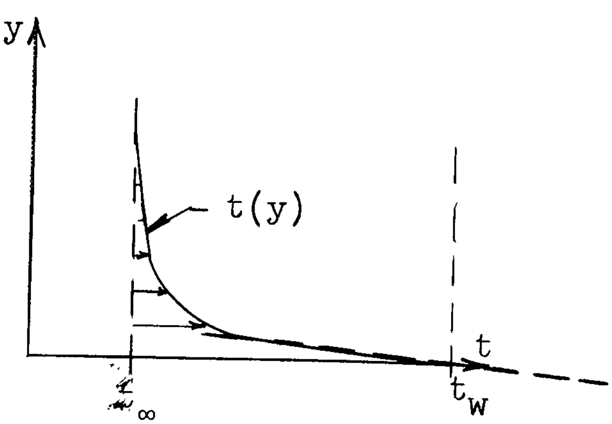
Before investigating the details of calculation of heat-transfer rates from nonisothermal surfaces, it is desirable to obtain a qualitative understanding of the way nonisothermality might be expected to influence the heat transfer. This can be best achieved by examining the "history" of the boundary layer and considering the qualitative effect of events upstream on the temperature profile in the boundary layer. While these remarks apply to a heated plate, they may readily be extended to a cooled plate.

If the plate is at constant temperature, the temperature profiles at different points on the plate will have the same general shape and will appear as shown in sketch (a). The slope at the wall will be proportional to the heat-transfer rate.



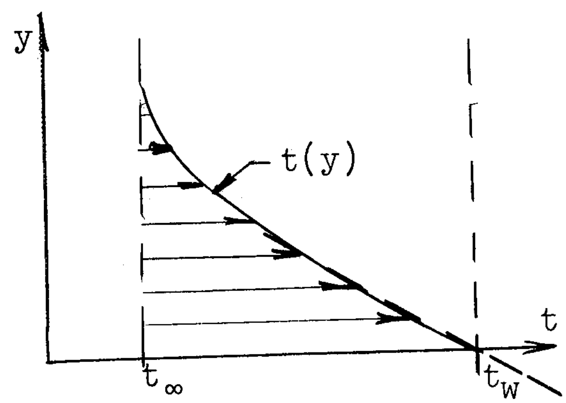
(a)

If the plate temperature increases in the flow direction, the temperature profiles will tend to be more drawn out, as indicated in sketch (b). The gradient at the wall will therefore be steeper, and the heat-transfer coefficient will be greater than if the plate were at constant temperature.



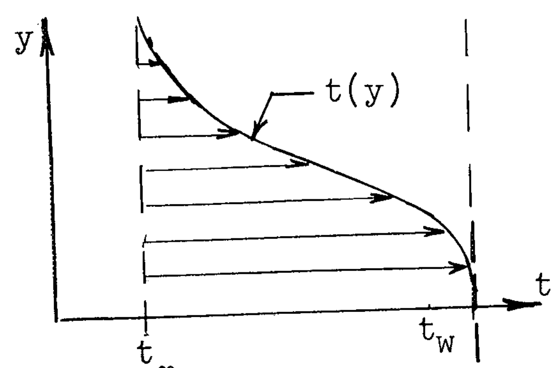
(b)

If, on the other hand, the wall temperature decreases in the flow direction, the profiles will tend to be less drawn out, as shown in sketch (c). The gradient at the wall will therefore be less, and the heat-transfer coefficient less than if the plate were at constant temperature.



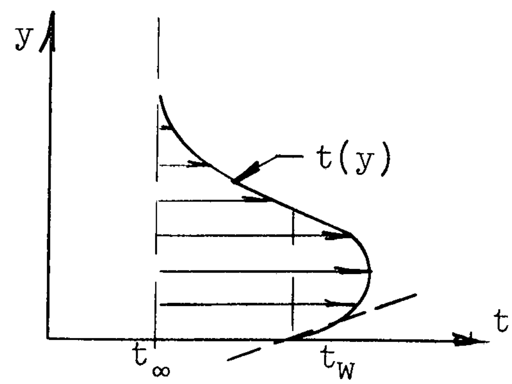
(c)

In fact, it is conceivable that, if the wall temperature decreased fast enough, the profile could be so distorted that the heat-transfer rate would be zero at some point, even with a finite over-all temperature difference. In such a case the temperature profile would appear as follows:



(d)

Moreover, decreasing the wall temperature even faster could result in a complete reversal of the profile near the wall (sketch (e)), in which case the heat transfer would be negative, with a positive over-all temperature difference.



(e)

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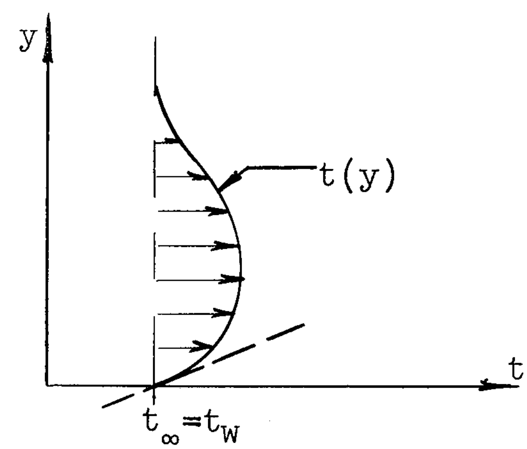
It is also possible that decreasing the temperature difference even faster could lead to finite negative heat transfer with a zero temperature difference:



(f)

It is evident from these considerations that the nonisothermality may have a profound influence on the shape of the temperature profiles in the boundary layer and consequently may greatly influence the heat transfer. Moreover, it is clear that the "history" of the boundary layer is of great importance. It should also be noted that the heat-transfer rate need not always be "in phase" with the wall-temperature variation. In general, however, the following is true (if $t_w > t_\infty$):

(1) A decreasing temperature difference leads to heat-transfer rates that are lower than those for an isothermal plate.

(2) An increasing temperature difference leads to heat-transfer rates that are higher than those for an isothermal plate.

With the effects of thermal history well in mind, the calculation of heat transfer from nonisothermal surfaces becomes purely a mathematical matter.

THEORY AND ANALYSIS

Review of Theory

The methods for determining heat transfer from nonisothermal surfaces are similar to the methods used in determining the deflection of beams subject to arbitrary load distributions. The energy equation of the boundary layer is linear in the fluid temperature if fluid properties are assumed to be constant. This allows superposition techniques to be employed. Rubesin (ref. 7) has shown that the heat-transfer rate for an arbitrary wall-temperature variation can be determined by superimposing a number of "step wall-temperature distributions," so that summation of the steps yields the actual variable temperature distribution, and the heat-transfer rate at any point is equal to the sum of the heat-transfer rates attributable to all "steps" upstream of the point in question. This idea of superposition is illustrated in figure 1, where the superposition of temperature steps is compared with the superposition of point loads used in beam-deflection problems. It is evident that a satisfactory solution for a step wall-temperature distribution is required before any attempt can be made to handle the variable-temperature problem, and such a solution for turbulent incompressible flow over a flat plate with a step wall-temperature distribution is presented in reference 4.

It is convenient at this point to introduce some new notation. Since the temperature distribution along the heated plate may be thought of as a function of the distance from the leading edge x, one may write

$$\Delta t = \Delta t(x)$$

On the other hand, the temperature distribution can always be represented by some algebraic expression involving x, and this expression may be a function of several parameters. Then one might prefer to denote the temperature difference more completely by

$$\Delta t = \Delta t(a_1, a_2, a_3, \ . \ . \ ., a_n; x)$$

where the a_i are the parameters in the functional description of the temperature difference. They might be the coefficients in a power series expansion, the locations of discontinuities, or some other parameters. Hereafter the parameters will always be listed before the important variable, which in this case is x. If only one symbol appears in the parentheses, it will refer to the important variable; and the fact that the function depends on the parameters a_1 to a_n is to be understood. Thus, for a step wall-temperature distribution, the heat-transfer coefficient may be written as $h(l;x)$, where l is the location of the discontinuity, or merely as $h(x)$, where it is to be understood from the context that the coefficient refers to the step distribution case.

It is also convenient to compare the local rate of heat transfer from a plate having variable wall temperature with the rate that will occur if the surface temperature is constant at its local value from the start of the plate. For this purpose it is convenient to denote the local Stanton number for an isothermal plate by St_T. The isothermal Stanton number is a function of x only and does not depend on any of the parameters characterizing the nonisothermal problem. The notation $St_T(x)$ will sometimes be used to emphasize this point. The isothermal Stanton number may be taken from any suitable expression for the Stanton number for heat transfer from a plate at constant temperature. For example, for heat transfer to a gas in the turbulent incompressible boundary layer, the isothermal Stanton number may be determined from

$$St_T(x) = 0.0296 Pr^{-0.4} Re_x^{-0.2} \left(\frac{T_W}{T_\infty}\right)^{-0.4} \tag{1}$$

Reference 3 shows that this relation is satisfactory for gases in the Reynolds number range $10^5 < Re_x < 10^7$. In using equation (1), the fluid properties in the Stanton, Prandtl, and Reynolds numbers are evaluated at the free-stream static temperature; the factor $(T_W/T_\infty)^{-0.4}$ provides the temperature-dependent fluid-property correction.

The step wall-temperature distribution case, which is the entire basis for superposition in arbitrary-wall-temperature problems, is treated in reference 4. The step wall-temperature distribution may be written as

$$\Delta t = \Delta t(l;x) = 0 \qquad x < l$$

$$\Delta t = \Delta t(l;x) = \Delta t_0 \qquad x > l$$

Reference 4 shows that the corresponding heat-transfer rate may be represented by

$$\frac{St(l;x)}{St_T(x)} = \left[1 - \left(\frac{l}{x}\right)^{9/10}\right]^{-1/9} \qquad x > l \tag{2}$$

The heat-transfer coefficient for the step case is therefore

$$h(l;x) = Gc_p St_T(x) \left[1 - \left(\frac{l}{x}\right)^{9/10}\right]^{-1/9} \tag{2a}$$

Following the methods of Klein and Tribus (ref. 6), one can superimpose an infinite number of small steps. This results in heat-transfer rates

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from the nonisothermal surface given by the following integral expression:

$$q''(x) = \int_{\xi=0}^{\xi=x} h(\xi;x)\ dt_w(\xi) \tag{3}$$

Here the kernel function $h(\xi,x)$ is, from equation (2a),

$$h(\xi;x) = Gc_p St_T(x)\left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-1/9} \tag{4}$$

Note that the terms $Gc_p St_T(x)$ may all be brought outside the integral in equation (3), since the integration is performed over the variable ξ. It should be noted that the integral of (3) must be taken in the "Stieltjes" sense (ref. 8) rather than in the ordinary "Riemann" or "area" sense. This must be done because the prescribed wall temperature may have a finite discontinuity, so that dt_w is undefined at some point. The Stieltjes integral may, however, be expressed as the sum of an ordinary or Reimann integral and a term that accounts for the effect of the finite discontinuities. The integral of equation (3) may be written as (ref. 8)

$$\underset{\text{(Stieltjes)}}{\int_{\xi=0}^{\xi=x} h(\xi;x)\ dt_w(\xi)} = \underset{\text{(Riemann)}}{\int_{\xi=0}^{\xi=x} h(\xi;x)\ \frac{dt_w(\xi)}{d\xi}\ d\xi} + \sum_{n=1}^{N} h(l_n;x)\left[t_w(l_n^+) - t_w(l_n^-)\right] \tag{5}$$

where $[t_w(l_n^+) - t_w(l_n^-)]$ denotes the temperature rise across the n^{th} discontinuity in the wall temperature. The use of equation (3) will be illustrated later by several examples.

Equation (3) is useful if the wall-temperature distribution is prescribed and the heat flux is to be calculated. An equally important problem is the case in which the heat flux is given and the wall temperature is to be found. Again following Klein and Tribus (ref. 6), the wall temperature may be determined from

$$\Delta t(x) = \int_{\xi=0}^{\xi=x} g(\xi;x)q''(\xi)\ d\xi \tag{6}$$

where

$$g(\xi;x) = \frac{(9/10)(1/x)}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T(x)}\left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} \tag{7}$$

Because of the nature of the integrand of equation (6), the integration can always be performed in the usual "Riemann" sense. The use of equation (6) in problems where the heat flux is prescribed will be illustrated later by examples.

Functions of Interest

In the solution of various arbitrary-wall-temperature problems, integrals are frequently encountered that cannot be integrated in closed form, but they are well known and may be determined from tables. Of particular interest here is the beta function, defined by (ref. 9)

$$B(a,b) = \int_0^1 z^{a-1}(1 - z)^{b-1}\,dz \tag{8a}$$

The beta function is a function of two arguments, a and b, and is not tabulated as such. It is, however, related to gamma functions by the relation (ref. 9)

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \tag{8b}$$

The gamma function is another integral but is a function of one argument only, and tables of this function are available (e.g., ref. 10). Note that $B(a,b) = B(b,a)$.

More properly, the beta function defined previously is referred to as the complete beta function. The incomplete beta function is also of interest, and it is defined by

$$B_r(a,b) = \int_0^r z^{a-1}(1 - z)^{b-1}\,dz \tag{8c}$$

Note that the complete beta function is $B_1(a,b)$. The incomplete beta function is tabulated, but not for arguments of interest in nonisothermal boundary-layer calculations. A number of incomplete beta functions of interest in turbulent heat-transfer analysis have been determined as a part of the current work. These functions are given in table I and are plotted in figure 2. In addition, the reader is referred to reference 11 for similar tabulations of incomplete beta functions for both laminar and turbulent flow over a flat plate. In calculating the

integrals use was made of the symmetry property of the incomplete beta function (ref. 12)

$$B_r(a,b) = B_1(a,b) - B_{1-r}(b,a) \tag{8d}$$

and the relation of the incomplete beta function to the hypergeometric function (ref. 12)

$$B_r(a,b) = \frac{r^a}{a} F(a,1-b;1+a;r) \tag{8e}$$

where the hypergeometric function is a well-known series. An IBM 650 computer was used to calculate the hypergeometric functions and the incomplete beta functions.

Some Special Nonisothermal Heat-Transfer Calculations

To illustrate the methods of superposition previously discussed, several examples of variable wall temperature and variable heat flux have been worked out in detail. These examples illustrate not only the methods of calculation, but also several interesting aspects of nonisothermal heat transfer. The results of these calculations are summarized in table II.

Constant heat input. - The temperature distribution along a flat plate at constant heat input may be determined from the theory of variable-wall-temperature heat transfer. The temperature distribution is, from equations (6) and (7),

$$\Delta t(x) = \frac{(9/10)q_0''}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T(x)} \int_{\xi=0}^{\xi=x} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\left(\frac{\xi}{x}\right) \tag{9a}$$

Setting $z = 1 - (\xi/x)^{9/10}$ reduces (9a) to

$$\Delta t(x) = \frac{q_0''}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T(x)} \int_0^1 z^{-8/9}(1 - z)^{1/9}\, dz \tag{9b}$$

The integral of (9b) is recognized as a complete beta function, and therefore

$$\Delta t(x) = \frac{q_0''B(1/9,10/9)}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T(x)} = 0.9587 \frac{q_0''}{Gc_pSt_T} \tag{10a}$$

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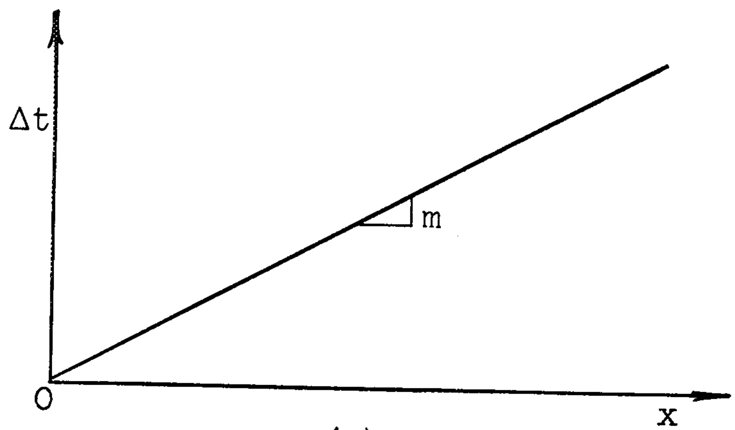
Putting (10a) into dimensionless form gives

$$\frac{St_H}{St_T} = 1.043 \tag{10b}$$

where St_H denotes the local Stanton number for constant heat input. This indicates that the local heat-transfer coefficient for constant heat input is 4.3 percent greater than that for constant wall temperature at any given location on the plate.

Ramp wall temperature. - Consider the "ramp" wall-temperature distribution as shown in sketch (g):



(g)

The wall temperature is given by $\Delta t(x) = mx$, where m is the slope of the wall temperature, $d\Delta t/dx$. Substituting in equation (3),

$$q''(x) = Gc_pSt_T(x)m \int_{\xi=0}^{\xi=x} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-1/9} d\xi \tag{11}$$

By letting $z = 1 - (\xi/x)^{9/10}$, the integral of (11) reduces to a complete beta function and leads to

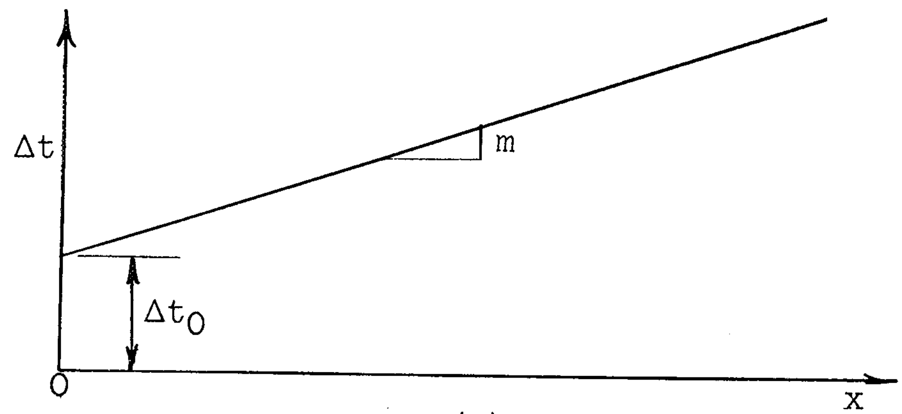
$$\begin{aligned} q''(x) &= Gc_pSt_T(x)mx(10/9)B(8/9,10/9) \\ &= 1.134mxGc_pSt_T(x) \end{aligned} \tag{12}$$

Since mx is simply the local temperature difference, equation (12) may be written as

$$\frac{St}{St_T} = 1.134 \tag{12a}$$

Hence, the heat-transfer rate is 13.4 percent greater than would be predicted from the equation for a constant-temperature plate.

Step-ramp wall temperature. - Consider the wall-temperature distribution indicated in sketch (h):



(h)

The heat-transfer rate for this wall-temperature distribution may be determined by superimposing the heat-transfer rates for a step at the leading edge and a ramp from the leading edge. Thus, from equations (4) and (12),

$$q''(x) = Gc_p St_T \Delta t_0 + 1.134 Gc_p St_T mx \tag{13}$$

which may be written as

$$\frac{St}{St_T} = \frac{1.134 + \Delta t_0/mx}{1 + \Delta t_0/mx} \tag{14}$$

Note that, as $\Delta t_0/mx$ approaches zero, the Stanton number for the combined step-ramp temperature distribution approaches that of the simple ramp (eq. (12a)), and as $\Delta t_0/mx$ becomes very large, the Stanton number approaches that of the isothermal plate. This is the expected limiting behavior.

The step-ramp example illustrates several interesting features of nonisothermal heat transfer that were mentioned earlier. For example, if the temperature difference increases with $x (m > 0)$, the heat-transfer rates will be higher than the isothermal rates. On the other hand, a decreasing temperature difference $(m < 0)$ leads to lower heat transfer; in fact, zero heat transfer may be obtained at some point with a finite temperature difference, and finite heat transfer at some point with a zero temperature difference. Physically, these situations arise because the energy and consequently the temperature in the boundary layer depend strongly upon the "history" of the layer. For example, if the

temperature difference decreases (the plate being heated), the fluid near the wall is hotter than it would have been if the plate temperature had been constant at its local value from the start. This means that the temperature gradient at the wall is less, and consequently the heat transfer is lower, than the isothermal values.

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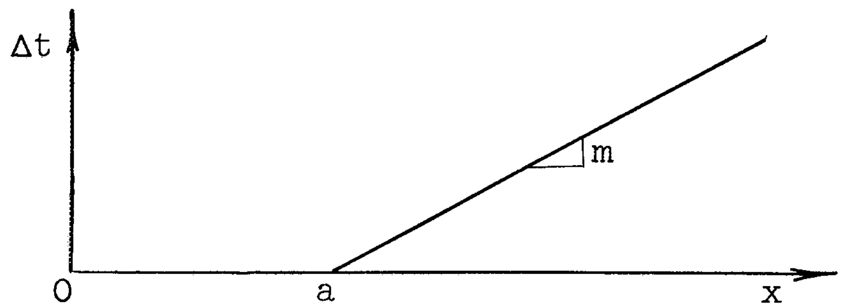
Delayed ramp wall temperature. - Another case of special interest is that where the temperature difference is zero over part of the plate and then varies linearly, as shown in sketch (i):



(i)

The temperature difference is given by

$$\Delta t = 0 \qquad x < a$$

$$\Delta t = m(x - a) \qquad x > a$$

Again using equation (3),

$$q''(x) = Gc_pSt_T(x)m \int_{\xi=a}^{\xi=x} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-1/9} d\xi \tag{15}$$

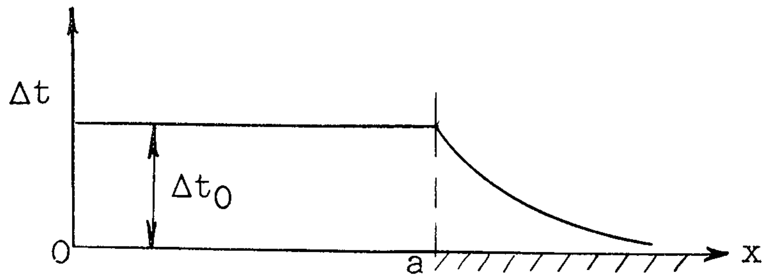
Letting $z = 1 - (\xi/x)^{9/10}$, the integral of equation (15) reduces to an incomplete beta function. This leads to the following result:

$$q''(x) = \frac{10}{9} Gc_pSt_T(x)mxB_r(8/9,10/9) \tag{16}$$

where $r = 1 - (a/x)^{9/10}$. This result is of utility in approximate solutions of variable-temperature problems, as is shown later.

Constant temperature followed by adiabatic wall. - Consider the case where the front part of the plate is held at constant temperature and the remainder of the plate is insulated, so that the heat transfer

is zero. The wall temperature is shown in sketch (j):



(j)

This case is of interest in the de-icing of aircraft wings, where the leading edge can be heated and the warm boundary layer used to melt the ice from the aft portion of the wing. Similar techniques would be useful in the cooling of high-speed aircraft and missiles.

This problem must be treated as a "specified heat input" problem. The heat flux over the isothermal portion can be computed from the definition of St_T:

$$q''(\xi) = Gc_p St_T(\xi)\Delta t_0 \qquad x < a \tag{17}$$

The heat input is zero for $x > a$. Then, using the fact that $St_T(\xi)$ varies as $\xi^{-0.2}$, the temperature of the adiabatic wall may be determined from equation (6) as

$$\Delta t(x) = \frac{(9/10)}{\Gamma(8/9)\Gamma(1/9)} \int_0^{a/x} \left(\frac{\xi}{x}\right)^{-0.2}\left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\left(\frac{\xi}{x}\right) \qquad x > a \tag{18}$$

By letting $z = (\xi/x)^{9/10}$, the integral of (18) can be reduced to an incomplete beta function, leading to

$$\frac{\Delta t}{\Delta t_0} = \frac{B_s(8/9,1/9)}{\Gamma(8/9)\Gamma(1/9)} \tag{19}$$

where $s = (a/x)^{9/10}$. It should be noted that there is no way to determine the "adiabatic decay temperature" (Δt) from the isothermal heat-transfer equation, and that nonisothermal theory is essential in the solution of this problem.

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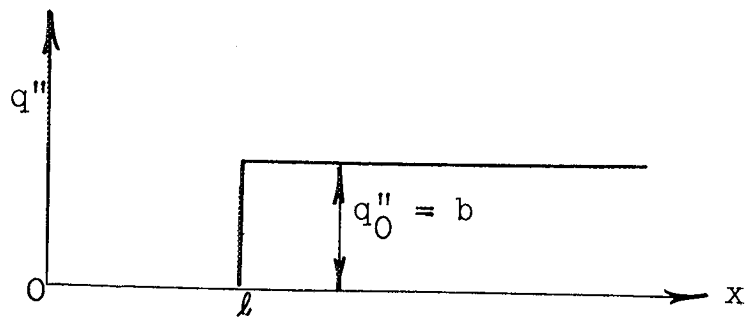
Step heat input. - Consider a step heat input as indicated in sketch (k):



(k)

The wall-temperature distribution rate may be determined from equation (6) as follows:

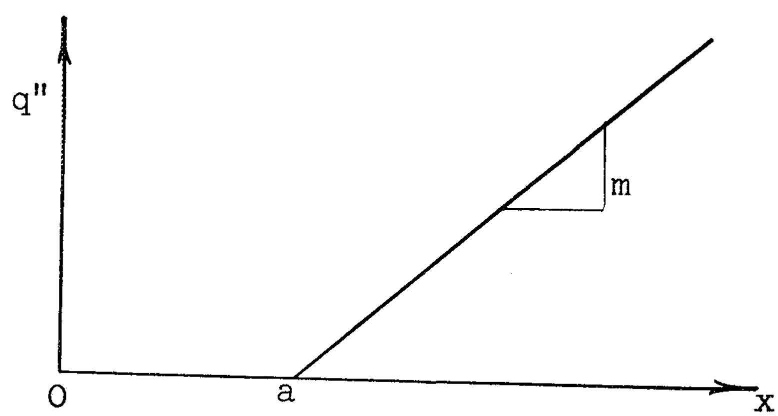
$$\Delta t(x) = \frac{(9/10)q_0''}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T(x)} \int_{l/x}^{1} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} d\left(\frac{\xi}{x}\right) \qquad (20a)$$

By letting $z = 1 - (\xi/x)^{9/10}$, equation (20a) integrates to give

$$\Delta t(x) = \frac{q_0''B_r(1/9,10/9)}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T(x)} \qquad (20b)$$

where $r = 1 - (l/x)^{9/10}$.

Delayed ramp heat input. - Consider a flat plate subjected to a "delayed ramp" heat input, as shown in sketch (l):



(l)

The temperature distribution for this specified-heat-input problem can be determined using equation (6), which leads to

$$\Delta t(x) = \frac{(9/10)mx}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T} \int_{a/x}^{1} \left[1 - \left(\frac{\xi}{x}\right)^{9/10}\right]^{-8/9} \left(\frac{\xi}{x} - \frac{a}{x}\right) d\left(\frac{\xi}{x}\right)$$

which, by appropriate substitution, reduces to

$$\Delta t(x) = \frac{mx}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T} \left[B_r(1/9,20/9) - \left(\frac{a}{x}\right)B_r(1/9,10/9)\right] \qquad (21)$$

where $r = 1 - (a/x)^{9/10}$. This result is also of interest in the approximate treatment of variable-heat-flux problems, as is described later.

Approximate Methods for Handling Problems of Variable Wall Temperature and Heat Flux

In many cases the prescribed wall-temperature distribution or heat flux will be such that the integrals resulting from the superposition methods cannot readily be evaluated in closed form. In such cases a long and tedious numerical integration is required for each point at which the heat transfer is to be calculated. It is evident that a suitable approximate technique for rapid calculations would be extremely useful. One possible method is to superimpose a finite number of steps to approximate the wall-temperature or heat-flux distribution. However, this leads to infinite heat-transfer rates at the discontinuities. A more satisfactory method is to superimpose a number of "ramps" to approximate the temperature or heat-flux distribution. Generally, a relatively few ramps can be used to obtain an excellent approximation to any type of prescribed distribution. Furthermore, since no discontinuities in temperature occur, no infinities in the heat-transfer rates will be obtained when ramps are used. The method of superposition of a number of delayed ramps is indicated by figure 3. Note that each ramp extends indefinitely downstream from the point of its origin. Any discontinuities in the prescribed distribution can be accounted for by adding a step, as is indicated by figure 3. Numerical examples of this method are worked out and compared with experiment later.

The method indicated is similar to that described recently in reference 13, in which some integrals of interest in laminar and turbulent variable-temperature problems are computed and a concise computation procedure is presented. Computations are given for the result of both Rubesin (ref. 7) and Seban (ref. 14), the latter of which is similar to the result of the present investigation. The present scheme has the advantage that

the computing equations have been put in a form where loss of significant figures due to subtraction of numbers of the same order of magnitude is minimized. Moreover, the present method allows steps to be used in addition to ramps, while in the method of reference 13 the distribution must be approximated entirely by ramps. Moreover, reference 13 treats only problems of prescribed wall temperature, while the present work allows calculation of both prescribed temperature-distribution and prescribed heat-flux problems.

The approximation of the temperature or heat-flux distribution by ramps and steps may be represented mathematically by the expression

$$\left.\begin{matrix}\Delta t(x)\\ q''(x)\end{matrix}\right\} = \sum_{n=1}^{N} m_n(x - a_n) + \sum_{j=1}^{J} b_j \tag{22}$$

Here m_n is the slope of the n^{th} ramp and has the dimensions of $^{o}F/ft$ (or Btu/(hr)(cu ft)), and a_n is its starting point; b_j is the height of the j^{th} step and has the dimensions ^{o}F (or Btu/(hr)(sq ft)); N and J are the number of ramps and steps, respectively, which begin upstream of the given location x. Differentiation of equation (22) shows that the slope of the approximate distribution at any point is the sum of the slopes of all ramps starting upstream of that point. Thus, if $M(x)$ is the slope of the curve of temperature against distance at the point x,

$$\frac{dT}{dx}(x) = M(x) = \sum_{n=1}^{N} m_n \tag{23}$$

Equation (23) allows calculation of the slopes of the component ramps. The step height b_j is simply the rise across the step and may be calculated by a single subtraction. Thus, once the "break points" have been selected, ramps and steps may be drawn in and their parameters evaluated by a small amount of arithmetic.

<u>Prescribed temperature-difference problems</u>. - If the temperature difference is represented as the sum of a number of ramps and steps, the heat-transfer rate is simply the sum of the heat-transfer rates due to the component ramps and steps. Therefore, from equations (16) and (4),

$$q''(x) = (10/9)Gc_pSt_T(x)x \sum_{n=1}^{N} m_n B_{r_n}(8/9,10/9)$$
$$+ Gc_pSt_T(x) \sum_{j=1}^{J} b_j\left[1 - \left(\frac{l_j}{x}\right)^{9/10}\right]^{-1/9} \tag{24}$$

where l_j is the location of the j^{th} step, and $r_n = 1 - (a_n/x)^{9/10}$. For calculation purposes, it is convenient to rewrite equation (24) in a different form. In order to do this, two new functions are defined:

$$A\left(\frac{a}{x}\right) = (10/9)B_r(8/9,10/9) - \left(1 - \frac{a}{x}\right) \tag{25}$$

and

$$S\left(\frac{l}{x}\right) = \left[1 - \left(\frac{l}{x}\right)^{9/10}\right]^{-1/9} - 1 \tag{26}$$

Then, by using equation (22), equation (24) becomes

$$q''(x) = Gc_pSt_T(x)\left[\Delta t(x) + x\sum_{n=1}^{N} m_nA\left(\frac{a_n}{x}\right) + \sum_{j=1}^{J} b_jS\left(\frac{l_j}{x}\right)\right] \tag{27}$$

This form of the computing equation has the advantage that the loss of significant figures due to subtraction of numbers of the same order of magnitude is relatively small. It also allows direct comparison with what would be predicted from the isothermal-plate relation (eq. (2)), namely

$$q_T'' = Gc_pSt_T(x)\Delta t(x) \tag{28}$$

The effect of the nonisothermality is then concentrated entirely in the summation terms. In order to simplify the form, an "effective temperature difference" Δt^* can be defined as

$$\Delta t^*(x) = \Delta t(x) + x\sum_{n=1}^{N} m_nA\left(\frac{a_n}{x}\right) + \sum_{j=1}^{J} b_jS\left(\frac{l_j}{x}\right) \tag{29}$$

This allows equation (27) to be written as

$$q''(x) = Gc_pSt_T(x)\Delta t^* \tag{30}$$

The relations developed allow rapid calculation of heat-transfer rates in prescribed wall-temperature-difference problems. The computing functions, A and S, are tabulated in table III and shown in figure 4. An example of calculation of the heat-transfer rate for a prescribed temperature distribution is presented later.

Prescribed heat-flux problems. - Prescribed heat-flux problems may be handled in the same manner as prescribed temperature-difference

problems, making use of the solution for a delayed ramp heat flux (22). Denoting $q''_n(x)$ as the heat flux due to the n^{th} ramp and $q''_j(x)$ as that due to the j^{th} step heat input, equation (6) may be written as

$$\Delta t(x) = \int_{\xi=0}^{\xi=x} g(\xi,x)\left[\sum_{n=0}^{N} q''_n(\xi) + \sum_{j=0}^{J} q''_j(\xi)\right] d\xi \tag{31}$$

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Exchanging the order of integration and summation gives

$$\Delta t(x) = \sum_{n=0}^{N} \int_{\xi=0}^{\xi=x} g(\xi,x)q''_n(\xi)d\xi + \sum_{j=0}^{J} \int_{\xi=0}^{\xi=x} g(\xi,x)q''_j(\xi)d\xi \tag{32}$$

But the integrals of (32) are merely those for single ramps and steps; therefore, from (20) and (22),

$$\Delta t(x) = \frac{1}{\Gamma(1/9)\Gamma(8/9)Gc_pSt_T}\left\{\sum_{n=0}^{N} m_n x\left[B_{r_n}(1/9,20/9) - \left(\frac{a_n}{x}\right)B_{r_n}(1/9,10/9)\right] + \sum_{j=0}^{J} b_j B_{r_j}(1/9,10/9)\right\} \tag{33}$$

where $r_n = 1 - (a_n/x)^{9/10}$, and $r_j = 1 - (l_j/x)^{9/10}$. Equation (33) may be put in a more useful form for computation by introducing two new functions:

$$D\left(\frac{a}{x}\right) = \left(1 - \frac{a}{x}\right) - \frac{B_r(1/9,20/9) - \left(\frac{a}{x}\right)B_r(1/9,10/9)}{\Gamma(1/9)\Gamma(8/9)} \tag{34}$$

and

$$H\left(\frac{l}{x}\right) = 1 - \frac{B_r(1/9,10/9)}{\Gamma(1/9)\Gamma(8/9)} \tag{35}$$

Then, using equation (22), equation (33) may be written as

$$\Delta t(x) = \frac{q^*}{Gc_pSt_T(x)} \tag{36}$$

where

$$q^* = q''(x) - x \sum_{n=1}^{N} m_n D\left(\frac{a_n}{x}\right) - \sum_{j=1}^{J} b_j H\left(\frac{l_j}{x}\right) \tag{37}$$

and q^* represents an "effective heat-transfer rate," in which the effects of the nonisothermality are all concentrated in the summation terms. Equations (36) and (37) may be used to calculate temperature differences in prescribed heat-flux problems. A numerical example of the use of equation (37) is included later with experimental verification. The functions D and H are tabulated in table IV and shown by figure 5.

RESULTS AND DISCUSSION

Comparison of Theory and Experiment

Experiments have been performed for several cases of variable wall temperature and heat flux. The test apparatus consisted of a large heated plate with an active flow length of about 5 feet built up with 24 individually heated strips. Within control limitations, any desired wall-temperature or heat-flux distribution could be obtained. The plate was tested in the 7-foot-diameter free-jet Guggenheim wind tunnel at Stanford University. Air velocities up to 130 feet per second were employed. This apparatus is described in detail in reference 3.

The data from these tests have been compared with predictions obtained from nonisothermal theory. In all of these predictions, the isothermal Stanton number St_T was computed from the relation

$$St_T Pr^{0.4} = 0.0296 Re_x^{-0.2} \left(\frac{T_w}{T_\infty}\right)^{-0.4} \tag{38}$$

In using this relation, all fluid properties are evaluated at the free-stream static temperature, and the factor $(T_w/T_\infty)^{-0.4}$ is a temperature-dependent fluid-properties correction. For comparison, predictions of the behavior of the nonisothermal surfaces have been made employing both the step-function analysis presented in reference 4 and an earlier analysis due to Rubesin (ref. 7).

Constant heat input. - Four test runs were made in which the heat flux was held constant. These data are tabulated in table V(a) and are shown on figure 6. The data are compared with the isothermal-plate correlation (eq. (38)), the constant-heat-input predictions of the present analysis (eq. (10b)), and the constant-heat-input prediction of Rubesin (ref. 7). Since the Stanton number for a surface with constant heat

input is only about 4 percent greater than that for constant wall temperature, the experimental uncertainty largely masks the predicted difference. At the start of the plate the data appear to be closer to the constant-temperature correlation; this is because the strips are long relative to the flow length, and each strip is essentially at constant temperature. Farther downstream, where the strip length is small compared with the flow length, the data agree very well with the constant-heat-input prediction,

$$St_H Pr^{0.4} = 0.0307 Re_x^{-0.2} \left(\frac{T_w}{T_\infty}\right)^{-0.4} \tag{39}$$

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The prediction of Rubesin ($St_H/St_T = 1.06$) appears to be slightly high over the entire test range.

Double-step wall temperature. - One test run was made for which the wall-temperature distribution was a "double step," a discontinuity in the wall temperature occurring in the center of the plate. These data are tabulated in table V(b) and shown in figure 7. Over the first portion of the plate, where the plate is at constant temperature, the predictions of the present analysis and of Rubesin (ref. 7) are identical. After the discontinuity, Rubesin's analysis is high, while the prediction of the present analysis is quite good. The predicted heat-transfer rates were determined by adding the rates due to the step at the leading edge to those due to the step downstream.

Step-ramp wall temperature. - One test run was made in which the wall temperature was varied linearly from the leading edge. Because of control limitations, it was necessary to have a small step in the wall temperature at the leading edge. The data from this run are tabulated in table V(c) and shown in figure 8. The linearity of the wall temperature was quite good. The heat-transfer data are compared with the predictions from the present analysis, with the result of Rubesin (ref. 7), and with what would be obtained from use of the uncorrected isothermal-plate correlation (eq. (38)), employing the local temperature difference. The predictions of the present analysis, which were determined from equation (13), agree very well with the experimental values. The heat-flux predictions based on the analysis of Rubesin are high, while the isothermal relation (38) predicts heat fluxes that are slightly low. However, the ramp obtainable with the experimental apparatus was not very steep, and thus the effect of the nonisothermality is not as great as might be obtained (see eq. (14)). In fact, use of the isothermal equation (38) and the local temperature differences gives results that may well be satisfactory as first approximations for engineering design calculations.

Constant temperature followed by adiabatic wall. - One test run was made in which the forward part of the plate was held at constant temperature and the power was turned off on the last 12 strips. The data from

this run are tabulated in table V(d) and shown by figure 9. In this example, the hot air flowing over the unheated portion of the plate warms the plate, and the result of interest is the wall-temperature distribution downstream of the discontinuity in heat flux. This situation is of interest in the de-icing of wings, where the leading edge may be heated and the hot air used to melt the ice from the after portion of the wing. Similar techniques are of interest in the cooling of missiles, where the front of the missile is cooled and the cold boundary layer is used to cool the rearward sections. The data are compared with the temperatures predicted by the present analysis (eq. (19)) and a similar equation obtained from the Rubesin analysis (ref. 7). Again the Rubesin prediction is high, while the agreement between the data and the present analysis is quite good. The measured temperatures are expected to lie slightly higher than the predictions because of conduction in the test plate. This conduction effect is greatest just downstream of the discontinuity in heat flux, and agreement of the data with the predicted temperature distribution is poorest in this region. The over-all agreement of the data with predictions is considered quite satisfactory.

Double-pulse heat input. - One test run was made with a double or step heat-flux distribution. The data from this run are tabulated in table V(e) and shown by figure 10. Because of conduction in the plate, it was not possible to obtain sharp discontinuities in the heat flux, as the data indicate. In making the predictions for the wall temperature, the actual nonperfect pulses were approximated by perfect pulses, so that the total heat transfer was the same. The approximate pulses are indicated in figure 10.

The predicted temperatures were determined by superposition of three steps of heat input, using the step heat-input results (eq. (20)). The agreement between the predicted and experimental temperatures is excellent, except near the ends of the pulses. The over-all agreement is considered quite satisfactory.

Irregular wall temperature. - One test run was made in which an extremely irregular wall-temperature distribution was maintained in the plate. The data from this run are tabulated in table V(f) and shown in figure 11(a).

The experimental heat-transfer rates are compared with heat-transfer rates predicted from the approximate methods described earlier. Figure 11(b) shows how the wall-temperature distribution was approximated by seven ramps and one step. Again the predicted heat-transfer rates are in excellent agreement with the predictions of the approximate method. The data for strip 2 are low, probably because the flow was not fully turbulent over this strip. The heat-transfer rate predicted by use of the isothermal-plate relation (38) is in considerable error, especially

where the wall temperature is changing rapidly. However, the values are of the same order of magnitude, and it can therefore be concluded that the isothermal equation may be used to obtain a first estimate of the heat transfer from nonisothermal surfaces.

This same example has also been treated as a "prescribed-heat-flux" problem, using the approximate methods described earlier. Figure 12(a) shows how the heat flux was approximated by five steps and six ramps. Figure 12(b) shows the temperature differences computed by the approximate methods compared with the experimental temperatures. The agreement near the leading edge is not too good, but this can be attributed to the approximations made on the heat-flux distribution, and to the fact that here the flow may not have been fully turbulent. Farther downstream, the predicted and experimental temperatures agree very well. The predictions made from the isothermal equation (38) are in considerable error, especially where the heat flux is changing rapidly. They are, however, of the same order of magnitude, and thus it appears that the isothermal equation can also be used to obtain a first approximation to the temperature difference in a prescribed heat-flux problem.

Determination of Heat-Transfer Rates by Approximate Methods

In order to illustrate the approximate methods for handling prescribed wall-temperature problems, the "irregular" example described in the preceding section will be worked out in detail. The test data for this run are shown in figure 11. The prescribed temperature difference will be approximated by seven ramps and one step, as is indicated by figure 11(a).

The first step is to determine the location and slopes of the ramps. The "break points" for the ramps and the temperature difference for the step will be taken as indicated in the following table, where n denotes the number of the ramp starting at a_n:

n	a_n, ft	$\Delta t(a_n)$, ^{o}F
1	0	4.0
2	.70	20.2
3	1.20	11.4
4	1.82	7.6
5	3.10	18.4
6	3.50	24.6
7	4.00	16.7
end	4.80	13.2

The slope of each component ramp may be determined readily, since the slope of the approximate temperature distribution at any point is simply the sum of the slopes of all ramps starting upstream. Therefore, for the first ramp,

$$m_1 = \frac{20.2 - 4.0}{0.70 - 0} = 23.14 \ {}^{\circ}F/ft$$

To find the slope of the second ramp,

$$m_1 + m_2 = \frac{11.4 - 20.2}{1.20 - 0.70} = -17.60 \ {}^{\circ}F/ft$$

$$m_2 = -17.60 - 23.14 = -40.74 \ {}^{\circ}F/ft$$

Similarly, for the third ramp,

$$m_1 + m_2 + m_3 = \frac{7.6 - 11.4}{1.82 - 1.20} = -6.13 \ {}^{\circ}F/ft$$

$$m_3 = -6.13 + 40.74 - 23.14 = 11.47 \ {}^{\circ}F/ft$$

For the fourth ramp,

$$m_4 + (-6.13) = \frac{18.4 - 7.6}{3.10 - 1.82} = 8.44 \ {}^{\circ}F/ft$$

$$m_4 = 14.57 \ {}^{\circ}F/ft$$

For the fifth ramp,

$$m_5 + 8.44 = \frac{24.6 - 18.4}{3.50 - 3.10} = 15.50 \ {}^{\circ}F/ft$$

$$m_5 = 7.06 \ {}^{\circ}F/ft$$

For the sixth ramp,

$$m_6 + 15.50 = \frac{16.7 - 24.6}{4.00 - 3.50} = -15.80 \ {}^{\circ}F/ft$$

$$m_6 = 31.30 \ {}^{\circ}F/ft$$

Finally, for the seventh ramp,

$$m_7 + (-15.80) = \frac{13.2 - 16.7}{4.80 - 4.00} = -4.37\ ^{\circ}F/ft$$

$$m_7 = 11.43\ ^{\circ}F/ft$$

In addition to the seven ramps, the superposition involves a step of 4.0° F at the leading edge, where $l_1 = 0$. Summarizing,

Ramps		
n	a_n, ft	m_n, °F/ft
1	0	23.14
2	.70	-40.74
3	1.20	11.47
4	1.82	14.57
5	3.10	7.06
6	3.50	-31.30
7	4.00	11.43

Steps		
j	l_j, ft	b_j, °F
1	0	4.0

At this point a check should be made. For $4.00 < x < 4.80$ feet, the temperature distribution should be given by

$$\Delta t(x) = 4.0 + 23.14(x - 0) - 40.74(x - 0.70) + 11.47(x - 1.20)$$
$$+ 14.57(x - 1.82) + 7.06(x - 3.10) - 31.30(x - 3.50) + 11.43(x - 4.00)$$

By substituting $x = 4.80$ feet, the temperature at the "end" may be calculated from this relation as

$$\Delta t(4.80) = 4.0 + 111.07 - 166.93 + 41.29 + 43.42 + 12.00$$
$$-40.69 + 9.14 = 13.3^{\circ}\ F$$

This agrees with the value that exists at $x = 4.80$ feet, and therefore the calculated ramp slopes close with the proper value, providing a check.

The calculation of the heat-transfer rates can best be handled by tabular computation. For example, the calculation of the heat-transfer rate at $x = 3.5$ feet is as follows:

First, tabulate the parameters of interest. From the data,

$x = 3.5$ ft $\Delta t = 24.6^{\circ}$ F

$G = 32{,}400$ lb/(hr)(sq ft) $Pr = 0.70$

$\mu = 0.0439$ lb/(hr)(ft) $T_{\infty} = 528^{\circ}$ R

$c_p = 0.24$ Btu/(lb)($^{\circ}$F) $T_w = 553^{\circ}$ R

Next, in a tabular manner, calculate the sums appearing in equation (27):

n	a_n	a_n/x	$A(a_n/x)$	m_n	$A(a_n/x)m_n$
1	0	0	0.135	23.14	3.12
2	.70	.200	.130	-40.74	-5.30
3	1.20	.343	.124	11.47	1.42
4	1.82	.520	.112	14.57	1.63
5	3.10	.886	.053	7.06	.37
				$\sum A(a_n/x)m_n = 1.24\ ^{\circ}F/ft$	
j	l_j	l_j/x	$S(l_j/x)$	b_j	$S(l_j/x)b_j$
1	0	0	0	4.0	0
				$\sum S(l_j/x)b_j = 0^{\circ}$ F	

Note that only the ramps and steps starting upstream of x are used in these calculations. The functions A and S are determined from figure 4, which may be replotted from the values in table III. The "effective" temperature in equation (29) is therefore

$$\Delta t^* = \Delta t(x) + x\sum m_n A(a_n/x) + \sum b_j S(l_j/x)$$

$$= 24.6 + 3.5(1.24) + 0$$

$$= 28.9^{\circ}\ F$$

The Reynolds number is

$$Re_x = Gx/\mu = (32{,}300)(3.5)/0.0439$$

$$= 2.58\times 10^6$$

The isothermal Stanton number St_T may be determined from equation (38) as

$$St_T = 0.0296(2.58\times10^6)^{-0.2}(0.7)^{-0.4}(553/528)^{-0.4}$$

$$= 0.00176$$

Finally, the heat-transfer rate may be calculated from equation (30) as

$$q''(3.5) = (32,300)(0.24)(0.00176)(28.9)$$

$$= 394 \text{ Btu/(hr)(sq ft)}$$

Similar calculations may be repeated at any point for which the heat-transfer rate is to be predicted. In making the predictions for this example, heat-transfer rates were calculated at the six "break points," the "end" (4.80 ft), and at $x = 0.4$ foot. The total calculating time was slightly less than 1 hour.

Determination of Wall Temperatures by Approximate Methods

In order to illustrate the approximate methods described earlier for handling prescribed heat-flux problems, the "irregular" example described earlier will be worked out in detail, assuming the heat flux is known and the wall temperature is to be calculated. The data from this run are shown in figure 12. The prescribed heat flux will be approximated by six ramps and five steps, as is indicated by figure 12(a). It will be assumed that the heat-transfer rate was constant from the leading edge, having a value equal to that measured for the second strip. This assumption will introduce some error at the start, but the boundary layer on the first strip is probably laminar or transitional, and thus a more accurate calculation is not possible.

The first step is to evaluate the parameters in the step-ramp approximation. The heat-flux discontinuities will occur midway between data points; at these points the two adjacent strips are separated by a thin insulator. The discontinuity and break-point locations are as follows:

x, ft	q"(x), Btu/(hr)(sq ft)
0-	0
0+	162.5
.46-	162.5
.46+	372.5
.90-	372.5
.90+	235.0
1.10-	235.0
1.10+	175.0
1.70	110.0
2.16-	110.0
2.16+	152.5
3.00	280.0
3.50	400.0
3.88	245.0
4.80	130.0

First, the height of each step is calculated as follows:

$$b_1 = 162.5 - 0 = 162.5 \text{ Btu/(hr)(sq ft)} \qquad l_1 = 0 \text{ ft}$$

$$b_2 = 372.5 - 162.5 = 210.0 \text{ Btu/(hr)(sq ft)} \qquad l_2 = 0.46 \text{ ft}$$

$$b_3 = 235.0 - 372.5 = -137.5 \text{ Btu/(hr)(sq ft)} \qquad l_3 = 0.90 \text{ ft}$$

$$b_4 = 175.0 - 235.0 = -60.0 \text{ Btu/(hr)(sq ft)} \qquad l_4 = 1.10 \text{ ft}$$

$$b_5 = 152.5 - 110.0 = 42.5 \text{ Btu/(hr)(sq ft)} \qquad l_5 = 2.16 \text{ ft}$$

The slope of each ramp is calculated from equation (23):

$$m_1 = (110.0 - 175.0)/(1.70 - 1.10)$$

$$= -108.3 \text{ Btu/(hr)(cu ft)}$$

$$m_2 + (-108.3) = (110.0 - 110.0)/(2.16 - 1.70) = 0$$

$$m_2 = +108.3 \text{ Btu/(hr)(cu ft)}$$

$$m_3 + 0 = (280.0 - 152.5)/(3.00 - 2.16) = 151.8$$

$$m_3 = 151.8 \text{ Btu/(hr)(cu ft)}$$

$$m_4 + 151.8 = (400.0 - 280.0)/(3.50 - 3.00) = 240.0$$

$$m_4 = 88.2 \text{ Btu/(hr)(cu ft)}$$

$$m_5 + 240.0 = (245.0 - 400.0)/(3.88 - 3.50) = -407.9$$

$$m_5 = -647.9 \text{ Btu/(hr)(cu ft)}$$

$$m_6 + (-407.9) = (130.0 - 245.0)/(4.80 - 3.88) = -125.0$$

$$m_6 = 282.9 \text{ Btu/(hr)(cu ft)}$$

The foregoing calculations may be summarized as follows:

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Ramps		
n	a_n, ft	m_n, Btu/(hr)(cu ft)
1	1.10	-108.3
2	1.70	108.3
3	2.16	151.8
4	3.00	88.2
5	3.50	-647.9
6	3.88	282.9

Steps		
j	l_j, ft	b_j, Btu/(hr)(sq ft)
1	0	162.5
2	.46	210.0
3	.90	-137.5
4	1.10	-60.0
5	2.16	42.5

At this point, it is desirable to make a check. At $x = 4.80$ feet, the heat flux is given by

$$
\begin{aligned}
q''(x) &= 162.5 + 210.0 - 137.5 - 60.0 + 42.5 - 108.3(4.80 - 1.10) \\
&\quad + 108.3(4.80 - 1.70) + 151.8(4.80 - 2.16) \\
&\quad + 88.2(4.80 - 3.00) - 647.9(4.80 - 3.50) + 282.9(4.80 - 3.88) \\
&= 130.0 \text{ Btu/(hr)(sq ft)}
\end{aligned}
$$

This result agrees with the value of q'' at the "end," and thus it appears that no errors have been made in determining the b and m.

The calculation for the temperature difference at various points on the plate can be handled in a tabular manner. To illustrate the method, the calculation for the temperature difference at $x = 3.5$ feet follows.

First, tabulate the parameters of interest. From the experimental data,

$x = 3.5$ ft $c_p = 0.24$ Btu/(lb)($^{\circ}$F)

$G = 32{,}400$ lb/(hr)(sq ft) $Pr = 0.70$

$\mu = 0.0439$ lb/(hr)(ft) $q'' = 400$ Btu/(hr)(sq ft)

Next, in a tabular manner, calculate the sums appearing in equation (37):

Steps					
j	l_j	l_j/x	$H(l_j/x)$	b_j	$b_j H(l_j/x)$
1	0	0	0.041	162.5	6.66
2	.46	.131	.055	210.0	11.55
3	.90	.257	.070	-137.5	-9.63
4	1.10	.314	.080	-60.0	-4.80
5	2.16	.617	.133	42.5	5.65
				$\sum b_j H(l_j/x) = 9.43$, Btu/(hr)(sq ft)	

Ramps					
n	a_n	a_n/x	$D(a_n/x)$	m_n	$m_n D(a_n/x)$
1	1.10	0.314	0.116	-108.3	-12.56
2	1.70	.486	.100	108.3	10.83
3	2.16	.617	.085	151.8	12.90
4	3.00	.857	.043	88.2	3.79
			$\sum m_n D(a_n/x) = 14.96$		
			$x \sum m_n D(a_n/x) = 52.36$ Btu/(hr)(sq ft)		

Therefore, the effective heat flux q^* is

$$q^* = 400.0 - 52.36 - 9.43$$

$$= 338.2 \text{ Btu/(hr)(sq ft)}$$

The Reynolds number is

$$Re_x = Gx/\mu = (32{,}300)(3.5)/0.0439$$

$$= 2.58\times 10^6$$

The isothermal Stanton number St_T may be determined from equation (38). However, since the temperature is not known, the fluid-properties correction factor $(T_W/T_\infty)^{-0.4}$ cannot be evaluated. This is, however, a small correction, which may be neglected without serious error. Thus,

$$St_T = 0.0296(2.58\times 10^6)^{-0.2}(0.7)^{-0.4}$$

$$= 0.00179$$

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Finally, the temperature difference may be calculated from equation (36):

$$\Delta t(3.5) = 338.2/(32,300)(0.24)(0.00179)$$

$$= 24.4^{\circ}\ F$$

Similar calculations may be repeated at each point for which the temperature difference is desired. One must be careful, however, at the steps. Just upstream of a step $(x = l_j^-)$, the step should not be included in the calculation. But just downstream of the step $(x = l_j^+)$, the step must be included, or else the computations would indicate a discontinuity in the wall temperature. Note that use of the isothermal equation (38) to predict the temperature leads to considerable error in heat flux at the steps where this relation indicated a discontinuity in the wall temperature. However, the isothermal equation is useful for an order-of-magnitude check on the calculations.

SUMMARY OF RESULTS

The good agreement between predicted and measured values of both heat-transfer rates and temperature distributions for the variety of situations shown is believed to substantiate the present theory for nonisothermal surfaces entirely adequately. Consequently, theory should be sufficiently accurate to predict heat transfer from nonisothermal surfaces in virtually any similar type of situation.

The results also show that the usual correlation for the heat-transfer rates, that in the solution for the isothermal surface, is in fair agreement with the data in all cases except where rapid changes in wall temperature or heat flux occur. Hence, in many cases the simpler isothermal equations can be used to obtain a first approximation for the heat-transfer rates (or temperature differences), even though the surface is not isothermal.

In cases where rapid changes in wall temperature occur, where a fraction of the surface is adiabatic, or where high accuracy is required, the nonisothermal theory should be used. In these situations, the approximate methods presented herein provide a simple, rapid computation method of good accuracy for the prediction of the temperature distribution or heat-transfer rates for any arbitrarily prescribed conditions.

Stanford University,
Stanford, Calif., October 22, 1957.

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TABLE I. - INCOMPLETE BETA FUNCTIONS FOR TURBULENT NONISOTHERMAL HEAT TRANSFER

$$\left[B_r(a,b) = \int_0^r z^{a-1}(1 - z)^{b-1}\, dz.\right]$$

r	$B_r(8/9,1/9)$	$B_r(8/9,10/9)$	$B_r(1/9,10/9)$	$B_r(1/9,20/9)$
0	0	0	0	0
.1	.1518	.1445	6.9604	6.8837
.2	.2948	.2661	7.5087	7.3445
.3	.4457	.3792	7.8447	7.5898
.4	.6109	.4863	8.0886	7.7414
.5	.7971	.5886	8.2796	7.8401
.6	1.0151	.6864	8.4357	7.9046
.7	1.2833	.7797	8.5664	7.9456
.8	1.6412	.8682	8.6772	7.9694
.9	2.2089	.9505	8.7702	7.9809
1.0	9.1853	1.0206	8.8439	7.9839

r	$B_r(1/9,8/9)$	$B_r(1/9,10/9)$	$B_r(1/9,20/9)$
0	0	0	0
.01	5.3960	5.3948	5.3887
.02	5.8286	5.8260	5.8131
.03	6.0979	6.0938	6.0735
.04	6.2967	6.2910	6.2631
.05	6.4555	6.4482	6.4125
.06	6.5883	6.5794	6.5357
.07	6.7029	6.6923	6.6406
.08	6.8039	6.7916	6.7316
.09	6.8943	6.8802	6.8119
.10	6.9764	6.9604	6.8837

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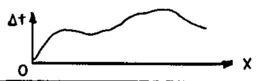
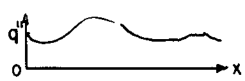
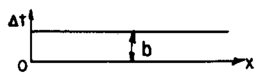
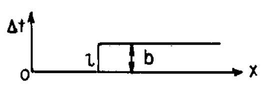
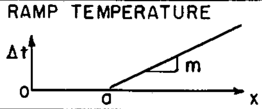
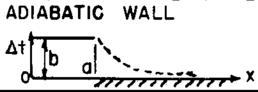
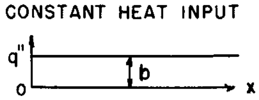
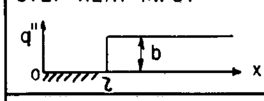
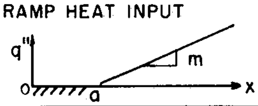
TABLE II. - SUMMARY OF TURBULENT NONISOTHERMAL HEAT-TRANSFER SOLUTIONS FOR A FLAT PLATE

SPECIFICATION	TEMPERATURE	HEAT FLUX
ARBITRARY TEMPERATURE 	$\Delta t = \Delta t(x)$	$q'' = Gc_p St_T \int_{\xi=0}^{\xi=x} \left[1-(\xi/x)^{9/10}\right]^{-1/9} d\Delta t(\xi)$
ARBITRARY HEAT FLUX 	$\Delta t = \frac{0.0975/x}{Gc_p St_T} \int_{\xi=0}^{\xi=x} \left[1-(\xi/x)^{9/10}\right]^{-8/9} q''(\xi)\, d\xi$	$q'' = q''(x)$
CONSTANT TEMPERATURE 	$\Delta t = b$	$q'' = Gc_p St_T\, b$
STEP TEMPERATURE 	$\Delta t = 0 \quad x \leq l$ $\Delta t = b \quad x \geq l$	$q'' = 0 \quad x < l$ $q'' = Gc_p St_T b\left[1-(l/x)^{9/10}\right]^{-1/9} \quad x \geq l$
RAMP TEMPERATURE 	$\Delta t = 0 \quad x < a$ $\Delta t = m(x-a) \quad x \geq a$	$q'' = 0 \quad x < a$ $q'' = 1.111\, mx\, Gc_p St_T\, B_r(8/9, 10/9) \quad x \geq a$
CONSTANT TEMPERATURE-ADIABATIC WALL 	$\Delta t = b \quad x < a$ $\Delta t = 0.1084\, b\, B_s(8/9, 1/9) \quad x \geq a$	$q'' = Gc_p St_T\, b \quad x < a$ $q'' = 0 \quad x \geq a$
CONSTANT HEAT INPUT 	$\Delta t = \frac{0.959\, b}{Gc_p St_T}$	$q'' = b$
STEP HEAT INPUT 	$\Delta t = 0 \quad x < l$ $\Delta t = \frac{0.959\, b}{Gc_p St_T} \frac{B_r(1/9, 10/9)}{B_1(1/9, 10/9)} \quad x \geq l$	$q'' = 0 \quad x < l$ $q'' = b \quad x \geq l$
RAMP HEAT INPUT 	$\Delta t = 0 \quad x < a$ $\Delta t = \frac{0.1084\, mx}{Gc_p St_T}\left[B_r(1/9, 20/9) - (a/x) B_r(1/9, 10/9)\right] \quad x \geq a$	$q'' = 0 \quad x < a$ $q'' = m(x-a) \quad x \geq a$

$St_T = 0.0296\, Pr^{-0.4}\, Re_x^{-0.2}$ $\qquad r = 1-(a/x)^{9/10} = 1-(l/x)^{9/10}$ $\qquad s = (a/x)^{9/10}$

TABLE III. - FUNCTIONS FOR PRESCRIBED

TEMPERATURE CALCULATIONS

a/x	$A(a/x)$	l/x	$S(l/x)$
0	0.135	0	0
.1	.133	.1	.015
.2	.130	.2	.030
.3	.126	.3	.047
.4	.121	.4	.066
.5	.113	.5	.089
.6	.103	.6	.116
.7	.100	.7	.152
.8	.074	.8	.208
.9	.050	.9	.300
1.0	0	1.0	∞

TABLE IV. - FUNCTIONS FOR PRESCRIBED

HEAT-FLUX CALCULATIONS

a/x	$D(a/x)$	l/x	$H(l/x)$
0	0.135	0	0.041
.1	.130	.1	.050
.2	.124	.2	.056
.3	.117	.3	.078
.4	.109	.4	.092
.5	.099	.5	.110
.6	.087	.6	.130
.7	.072	.7	.158
.8	.056	.8	.194
.9	.033	.9	.256
1.0	0	1.0	1.000

TABLE V. - EXPERIMENTAL HEAT-TRANSFER DATA

(a) Constant heat input

Strip	G, lb/(hr)(sq ft) $\times 10^{-3}$	Δt_m, °F	q", Btu/(hr)(sq ft)	h, Btu/(hr)(sq ft)(°F)	St $\times 10^3$	$St\left(\frac{T_w}{T_\infty}\right)^{0.4}$ $\times 10^3$	Re_x $\times 10^{-6}$	G, lb/(hr)(sq ft) $\times 10^{-3}$	Δt_m, °F	q", Btu/(hr)(sq ft)	h, Btu/(hr)(sq ft)(°F)	St $\times 10^3$	$St\left(\frac{T_w}{T_\infty}\right)^{0.4}$ $\times 10^3$	Re_x $\times 10^{-6}$
	t_∞ = 77.7° F; ρ_∞ = 0.0741 lb/cu ft							t_∞ = 79.1° F; ρ_∞ = 0.0741 lb/cu ft						
2	34.0	16.4	370	22.6	2.77	2.80	0.263	33.6	16.1	353	21.9	2.72	2.75	0.259
3	34.0	18.2	362	19.87	2.44	2.47	.431	33.7	17.6	350	19.86	2.46	2.49	.427
4	34.1	19.7	358	18.18	2.22	2.25	.593	33.4	19.1	344	18.01	2.25	2.28	.580
5	34.1	20.4	363	17.76	2.17	2.20	.753	33.6	19.8	351	17.71	2.20	2.24	.741
6	34.2	20.6	360	17.47	2.13	2.16	.915	33.5	20.0	347	17.35	2.16	2.19	.894
7	34.1	21.1	357	16.91	2.07	2.10	1.071	33.5	20.2	341	16.86	2.10	2.13	1.053
8	34.1	21.2	353	16.65	2.04	2.07	1.231	33.5	20.6	341	16.55	2.06	2.09	1.210
9	34.1	21.5	364	16.85	2.06	2.09	1.390	33.4	21.0	350	16.68	2.08	2.11	1.361
10	34.1	22.9	352	15.38	1.88	1.92	1.553	33.5	22.3	344	15.40	1.92	1.95	1.526
11	34.0	23.5	355	15.09	1.85	1.88	1.708	33.4	23.0	342	14.85	1.85	1.88	1.674
12	34.1	23.2	356	15.34	1.88	1.91	1.872	33.4	22.8	343	15.05	1.88	1.91	1.831
13	34.0	23.4	358	15.28	1.88	1.91	2.03	33.4	22.9	345	15.07	1.88	1.91	1.989
14	33.9	23.6	360	15.24	1.88	1.91	2.18	33.3	23.0	341	14.80	1.85	1.88	2.14
15	33.8	24.1	354	14.69	1.81	1.84	2.33	33.3	23.7	341	14.38	1.80	1.83	2.29
16	33.8	24.2	355	14.68	1.81	1.85	2.49	33.3	23.7	343	14.45	1.81	1.84	2.45
17	33.8	24.2	355	14.68	1.81	1.85	2.65	33.3	23.8	342	14.37	1.80	1.83	2.61
18	33.9	24.7	355	14.35	1.76	1.79	2.82	33.3	24.3	340	14.00	1.75	1.78	2.76
19	33.8	24.9	355	14.22	1.76	1.79	2.95	33.0	24.5	343	13.99	1.75	1.78	2.92
20	33.6	24.9	355	14.29	1.77	1.80	3.10	33.1	24.5	342	13.96	1.76	1.79	3.05
21	33.6	25.6	359	14.03	1.74	1.77	3.27	33.1	24.9	346	13.90	1.75	1.78	3.21
22	33.6	25.6	345	12.95	1.61	1.64	3.42	33.1	26.2	331	12.61	1.59	1.62	3.37
23	33.6	25.6	351	13.20	1.64	1.67	3.58	33.1	26.0	342	13.15	1.66	1.69	3.52
	t_∞ = 77.5° F; ρ_∞ = 0.0741 lb/cu ft							t_∞ = 78.9° F; ρ_∞ = 0.0740 lb/cu ft						
2	31.4	18.2	370	20.3	2.69	2.73	0.243	30.2	18.3	350	19.18	2.64	2.68	0.233
3	31.5	19.4	372	19.16	2.54	2.57	.400	30.4	19.1	356	18.66	2.56	2.59	.385
4	31.7	21.0	364	17.34	2.28	2.31	.551	30.5	20.5	352	17.15	2.34	2.38	.529
5	31.6	21.7	370	17.03	2.25	2.28	.698	30.6	21.3	356	16.83	2.30	2.33	.674
6	31.7	22.1	368	16.65	2.19	2.23	.848	30.6	21.6	353	16.34	2.29	2.26	.816
7	31.6	22.5	364	16.18	2.14	2.17	.993	30.5	22.1	350	15.82	2.16	2.19	.958
8	31.5	22.8	362	15.86	2.10	2.13	1.139	30.5	22.6	345	15.24	2.08	2.12	1.101
9	31.5	23.0	369	16.03	2.12	2.16	1.287	30.5	22.7	354	15.62	2.14	2.17	1.244
10	31.5	24.3	363	14.92	1.98	2.01	1.435	30.5	24.0	348	14.51	1.98	2.02	1.388
11	31.5	24.9	360	14.45	1.91	1.95	1.582	30.4	24.6	347	14.09	1.93	1.97	1.522
12	31.5	24.8	368	14.84	1.96	2.00	1.730	30.4	24.7	351	14.22	1.95	1.99	1.668
13	31.6	24.9	361	14.50	1.91	1.95	1.888	30.4	24.7	346	14.00	1.92	1.95	1.811
14	31.6	25.1	364	14.50	1.91	1.95	2.04	30.4	24.9	348	14.05	1.93	1.97	1.950
15	31.6	25.9	361	13.93	1.84	1.87	2.18	30.3	25.5	345	13.51	1.86	1.89	2.08
16	31.6	25.6	363	14.17	1.87	1.91	2.33	30.3	25.4	346	13.64	1.88	1.91	2.23
17	31.6	26.0	361	13.89	1.83	1.87	2.48	30.4	25.6	347	13.56	1.86	1.90	2.38
18	31.5	26.6	364	13.67	1.81	1.84	2.62	30.4	26.3	347	13.20	1.81	1.84	2.53
19	31.5	26.6	260	13.55	1.79	1.83	2.76	30.4	26.4	246	13.11	1.80	1.83	2.67
20	31.3	26.6	363	13.65	1.82	1.85	2.89	30.2	26.6	248	13.09	1.81	1.85	2.78
21	31.4	27.0	363	13.43	1.79	1.82	3.05	30.3	26.9	348	12.95	1.78	1.82	2.93
22	31.3	28.3	350	12.40	1.65	1.68	3.19	30.2	28.1	336	11.95	1.65	1.69	3.07
23	31.3	28.3	359	12.69	1.69	1.73	3.34	30.2	28.1	343	12.19	1.68	1.72	3.21

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TABLE V. - Concluded. EXPERIMENTAL HEAT-TRANSFER DATA

Strip	G, lb/(hr)(sq ft) $\times 10^{-3}$	Δt_m, °F	q", Btu/(hr)(sq ft)	h, Btu/(hr)(sq ft)(°F)	St $\times 10^3$	$St\left(\frac{T_w}{T_\infty}\right)^{0.4}$ $\times 10^3$	Re_x $\times 10^{-6}$
(b) Double-step temperature distribution [t_∞ = 72.5° F; ρ_∞ = 0.0746 lb/cu ft.]							
2	32.9	12.4	181.3	14.62	1.85	1.87	0.256
3	32.9	11.8	219	18.56	2.37	2.39	.421
4	32.9	11.8	207	17.57	2.23	2.25	.576
5	32.9	12.0	206	17.19	2.18	2.20	.731
6	32.9	12.1	204	16.88	2.14	2.16	.886
7	32.9	12.3	197.7	16.07	2.04	2.06	1.042
8	32.9	12.3	196.0	15.94	2.02	2.04	1.197
9	32.9	12.1	192.2	15.89	2.01	2.03	1.353
10	32.9	12.2	196.3	16.09	2.05	2.07	1.510
11	32.9	23.2	398	17.13	2.17	2.21	1.664
12	33.0	22.8	388	17.03	2.14	2.18	1.830
13	33.0	23.1	370	16.00	2.02	2.05	1.984
14	33.1	23.2	365	15.74	1.98	2.02	2.15
15	33.0	23.5	355	15.09	1.90	1.94	2.30
16	33.0	23.5	355	15.11	1.91	1.94	2.45
17	33.1	23.6	348	14.76	1.86	1.89	2.62
18	33.0	23.5	340	14.11	1.82	1.86	2.76
19	32.9	23.7	337	14.20	1.80	1.83	2.91
20	32.8	23.7	338	14.27	1.82	1.85	3.12
21	32.8	23.8	329	13.80	1.76	1.79	3.24
22	32.9	24.0	306	12.76	1.62	1.64	3.36
23	32.9	24.1	319	13.21	1.67	1.70	3.53
(c) Step-ramp temperature distribution [t_∞ = 73.3° F; ρ_∞ = 0.0749 lb/cu ft.]							
2	32.0	6.3	87.2	13.84	1.80	1.81	0.249
3	32.0	7.1	131.6	18.53	2.41	2.42	.410
4	32.0	7.9	144.4	18.28	2.38	2.39	.561
5	32.9	9.0	162.3	18.03	2.35	2.37	.712
6	32.0	9.9	178.2	18.00	2.34	2.36	.862
7	32.0	10.9	192.3	17.65	2.30	2.32	1.013
8	32.0	11.9	200	16.84	2.20	2.22	1.164
9	32.0	12.7	221	17.40	2.27	2.29	1.314
10	32.0	14.1	223	15.78	2.06	2.08	1.469
11	32.0	14.9	226	15.06	1.96	1.98	1.618
12	32.0	15.4	242	15.72	2.05	2.07	1.770
13	32.0	17.1	269	15.70	2.04	2.07	1.927
14	32.0	17.9	280	15.61	2.03	2.06	2.08
15	31.9	18.9	285	15.08	1.97	2.00	2.22
16	32.0	20.2	307	15.18	1.98	2.01	2.37
17	32.0	21.1	319	15.10	1.96	1.99	2.53
18	31.9	22.5	328	14.55	1.90	1.93	2.67
19	32.0	23.3	346	14.83	1.93	1.97	2.83
20	31.8	24.3	358	14.71	1.93	1.97	2.96
21	31.8	25.2	363	14.39	1.90	1.94	3.11
22	31.9	26.5	353	13.33	1.74	1.77	3.28
23	31.9	27.3	373	13.64	1.78	1.82	3.43
(d) Isothermal plate followed by adiabatic wall [t_∞ = 70.8° F; ρ_∞ = 0.0747 lb/cu ft.]							
2	27.0	26.0	228	9.77	1.51	1.54	0.211
3	27.0	25.7	366	14.22	2.19	2.23	.346
4	27.0	25.7	400	15.57	2.41	2.45	.473
5	27.0	25.7	386	15.00	2.32	2.36	.601
6	27.0	25.8	377	14.60	2.26	2.30	.728
7	27.0	25.8	364	14.10	2.17	2.21	.858
8	27.0	25.9	353	13.62	2.10	2.14	.985
9	27.0	26.0	357	13.71	2.11	2.15	1.117
10	27.0	26.3	333	12.66	1.95	1.99	1.243
11	27.0	26.4	320	12.12	1.87	1.90	1.370
12	27.0	26.4	327	12.38	1.91	1.94	1.498
13	27.0	11.6	43	-----	----	----	1.627
14	27.0	7.8	0	-----	----	----	1.756
15	27.0	6.3	0	-----	----	----	1.879
16	27.0	5.3	0	-----	----	----	2.02
17	27.0	4.9	0	-----	----	----	2.14
18	27.0	4.4	0	-----	----	----	2.27
19	27.0	4.0	0	-----	----	----	2.39
20	27.0	3.7	0	-----	----	----	2.52
21	27.0	3.4	0	-----	----	----	2.64
22	27.0	3.2	0	-----	----	----	2.77
23	27.0	3.0	0	-----	----	----	2.90
(e) Pulse heat input [t_∞ = 68.3° F; ρ_∞ = 0.0757 lb/cu ft.]							
2	32.8	0	0	-----	----	----	0.257
3	32.9	0	0	-----	----	----	.423
4	32.9	1.2	55.4	-----	----	----	.580
5	32.0	14.5	328	22.6	2.86	2.89	.739
6	33.0	17.2	367	21.3	2.69	2.73	.895
7	33.1	18.4	370	20.8	2.62	2.39	1.057
8	33.2	19.2	360	18.74	2.35	2.39	1.217
9	33.2	18.0	316	17.53	2.20	2.23	1.375
10	33.3	6.2	33.5	-----	----	----	1.540
11	33.2	3.6	0	-----	----	----	1.691
12	33.2	2.7	0	-----	----	----	1.850
13	33.3	2.1	0	-----	----	----	2.02
14	33.2	2.0	0	-----	----	----	2.17
15	33.3	3.0	47.5	-----	----	----	2.33
16	33.2	16.5	314	19.03	2.39	2.41	2.49
17	33.3	19.8	357	18.04	2.26	2.29	2.65
18	33.4	21.2	362	17.06	2.13	2.17	2.81
19	33.3	21.9	360	16.44	2.06	2.09	2.97
20	33.2	22.2	363	16.36	2.05	2.09	3.11
21	33.1	22.9	365	15.94	2.01	2.04	3.27
22	33.2	24.4	353	14.47	1.82	1.83	3.43
23	33.2	24.2	358	14.80	1.86	1.88	3.59
(f) Irregular wall-temperature distribution [t_∞ = 67.5° F; ρ_∞ = 0.0761 lb/cu ft.]							
2	32.2	10.4	164.8	15.84	2.05	2.07	0.252
3	32.2	17.7	368	20.8	2.69	2.73	.414
4	32.2	20.2	376	18.62	2.41	2.43	.567
5	32.2	14.2	235	16.56	2.14	2.17	.720
6	32.2	11.5	164.3	14.29	1.85	1.86	.874
7	32.2	10.0	130.0	13.00	1.68	1.69	1.028
8	32.3	8.7	117.2	13.47	1.74	1.75	1.181
9	32.3	8.0	105.4	13.18	1.70	1.71	1.338
10	32.2	8.8	112.6	12.79	1.66	1.67	1.488
11	32.3	11.0	169.2	15.38	1.98	2.00	1.647
12	32.2	17.5	191.1	15.28	1.97	1.99	1.794
13	32.3	14.7	236	16.02	2.06	2.09	1.957
14	32.3	16.5	266	16.10	2.08	2.11	2.10
15	32.3	18.3	287	15.69	2.02	2.05	2.26
16	32.4	21.8	354	16.22	2.09	2.16	2.42
17	32.4	24.6	398	16.17	2.08	2.12	2.58
18	32.3	22.0	313	14.21	1.83	1.87	2.72
19	32.3	17.9	232	12.94	1.67	1.69	2.88
20	32.1	16.1	208	12.89	1.67	1.69	3.01
21	32.1	15.5	188	12.10	1.57	1.59	3.16
22	32.3	14.4	154	10.70	1.38	1.40	3.33
23	32.3	12.4	126	10.18	1.32	1.33	3.45

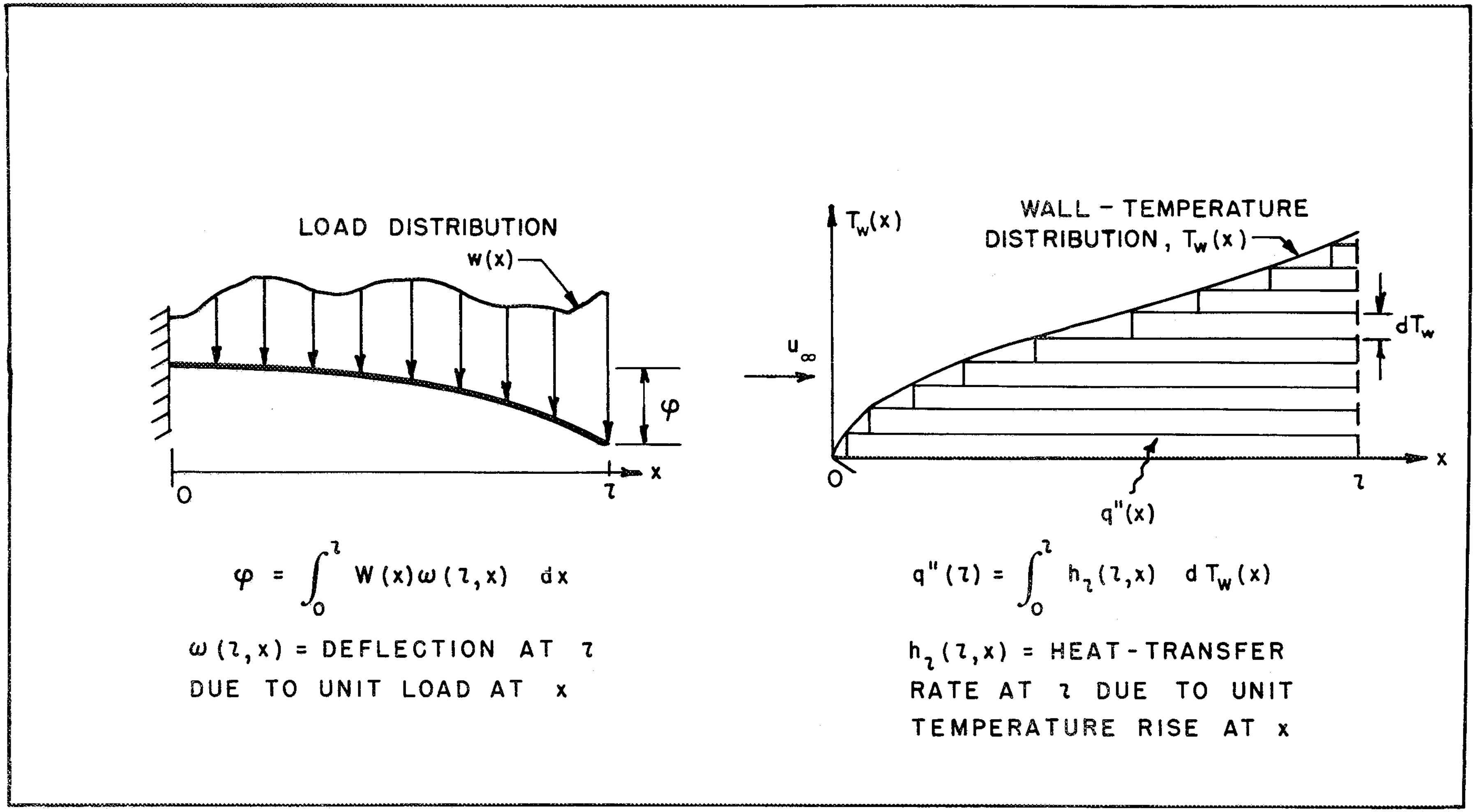

Figure 1. - Analogy of superposition techniques for beam deflection and heat transfer.

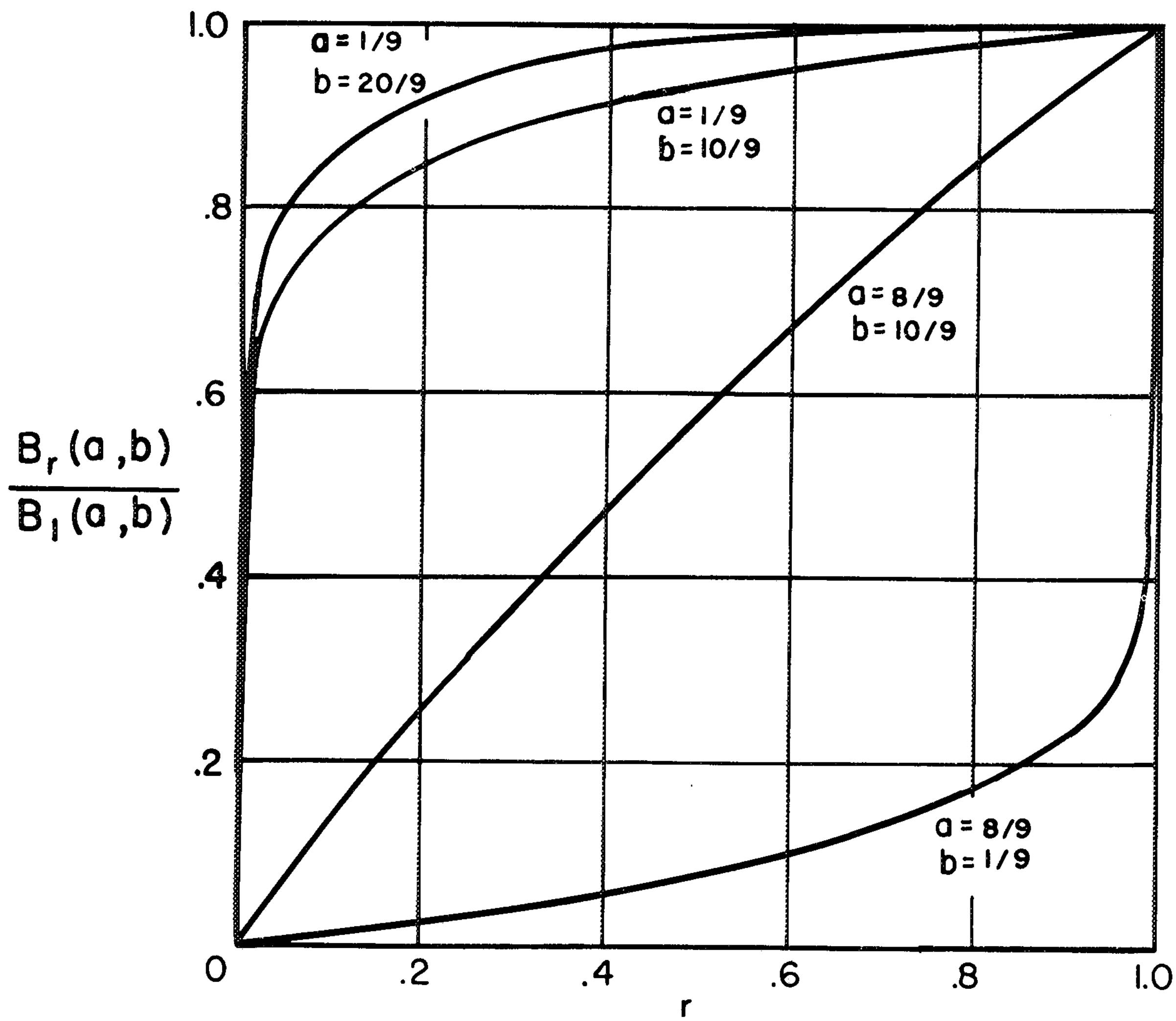

Figure 2. - Incomplete beta functions for turbulent nonisothermal heat transfer. Beta function $B_r(a,b) = \int_0^r z^{a-1}(1 - z)^{b-1}\, dz$.

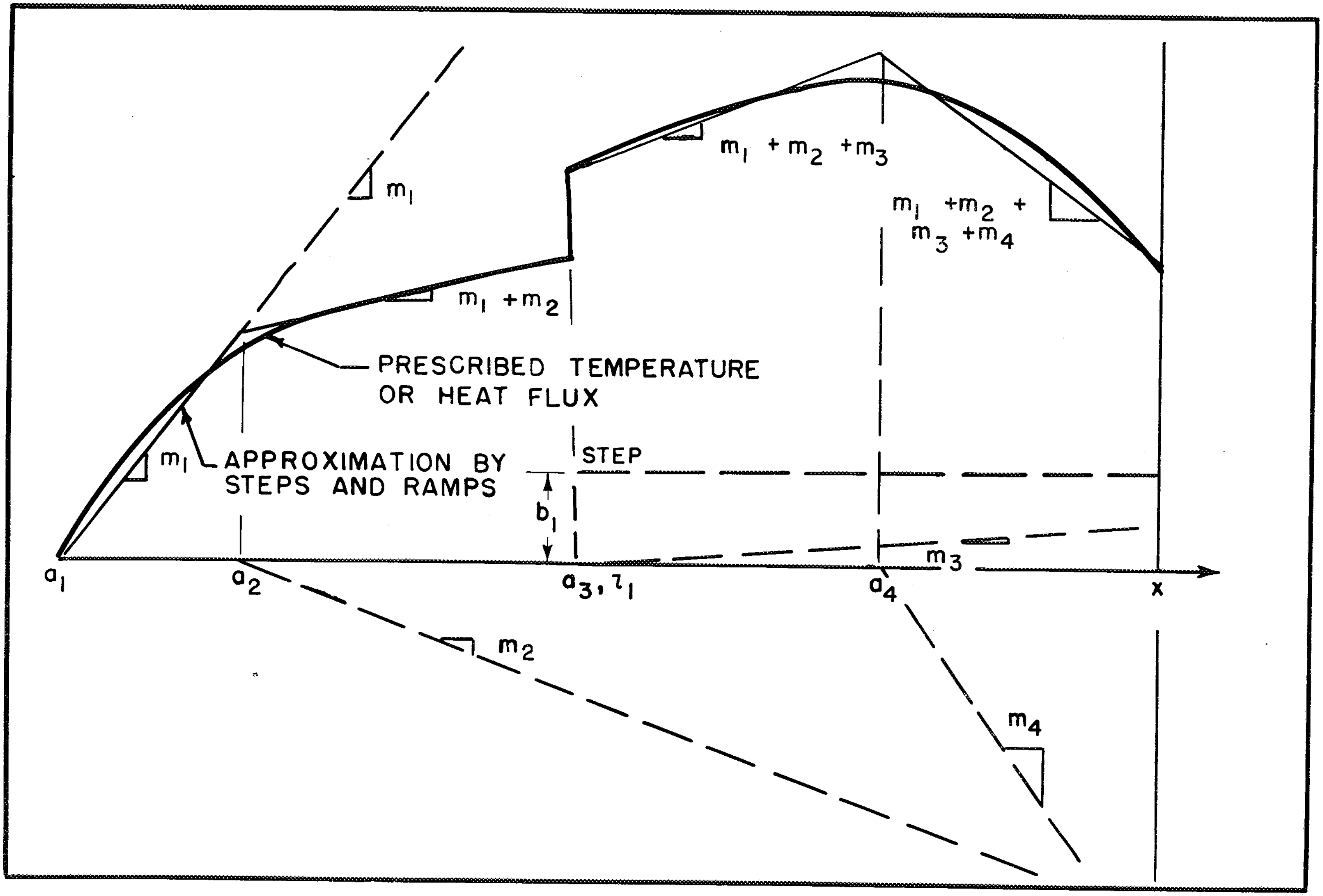

Figure 3. - Approximation of prescribed temperatures or heat flux by steps and ramps.

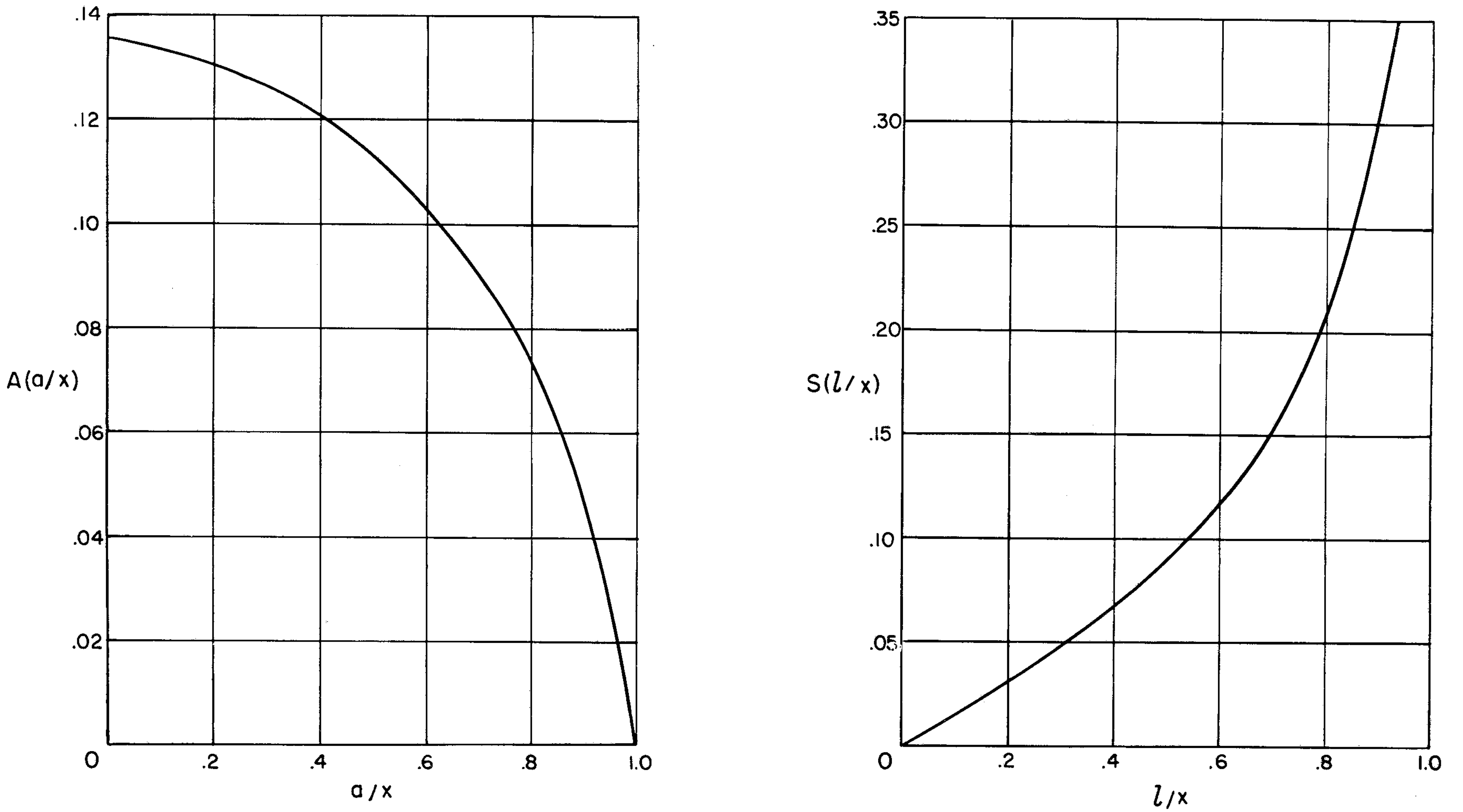

Figure 4. - Functions for prescribed temperature problems.

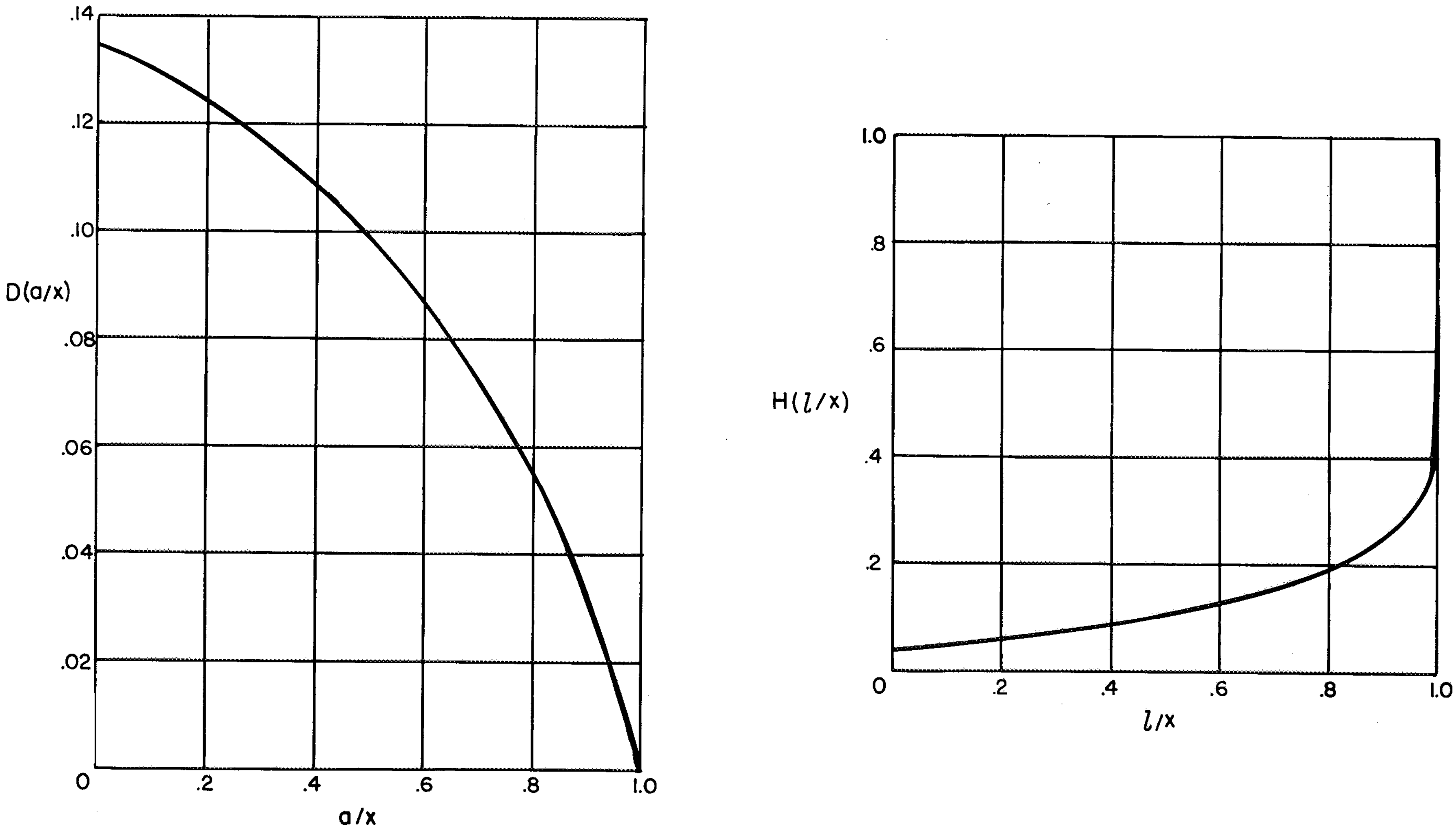

Figure 5. - Functions for prescribed heat-flux problems.

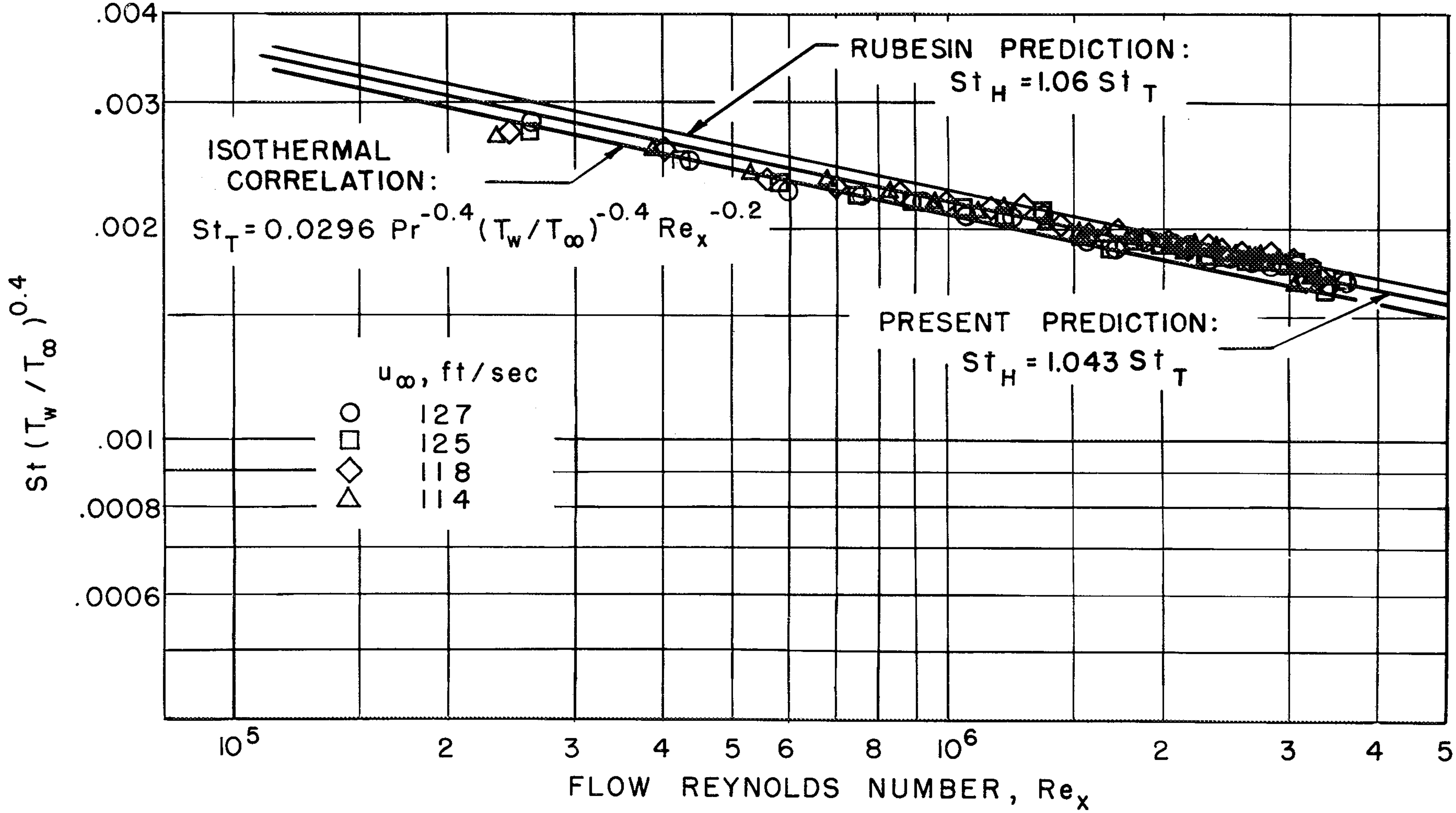

Figure 6. - Local Stanton numbers for constant heat input. Prandtl number, 0.7.

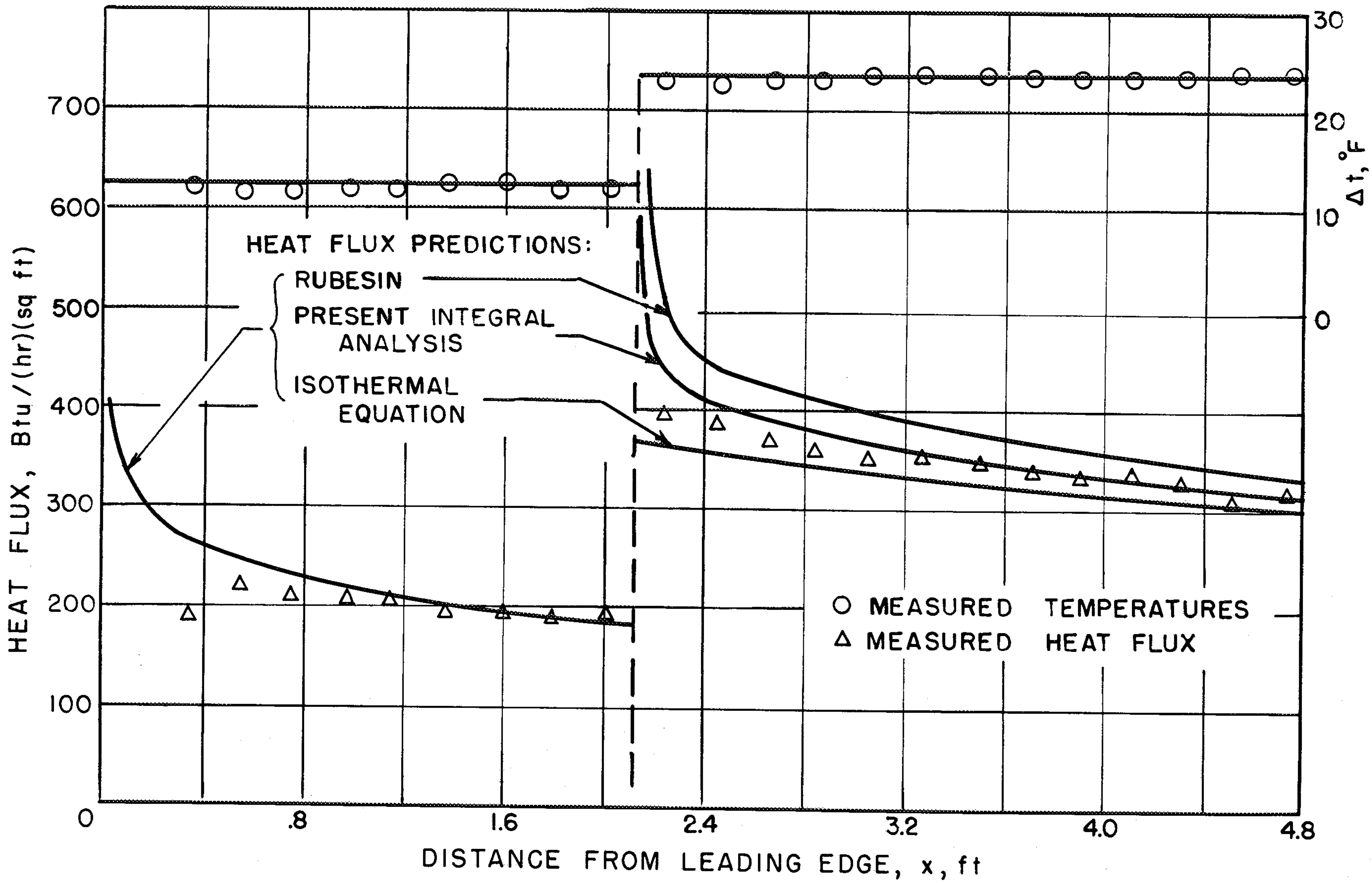

Figure 7. - Example for double-step temperature distribution.

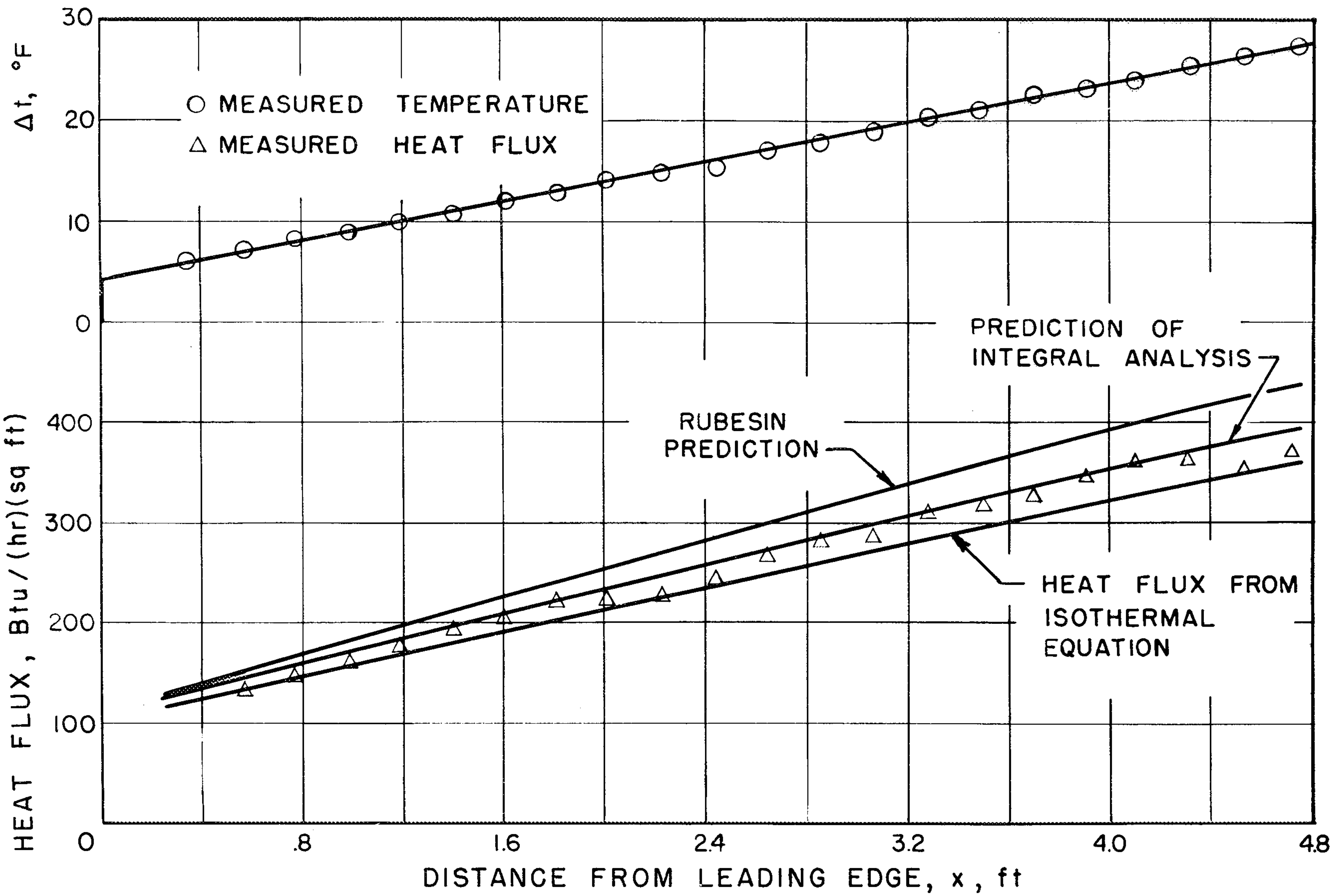

Figure 8. - Example for step-ramp wall temperature.

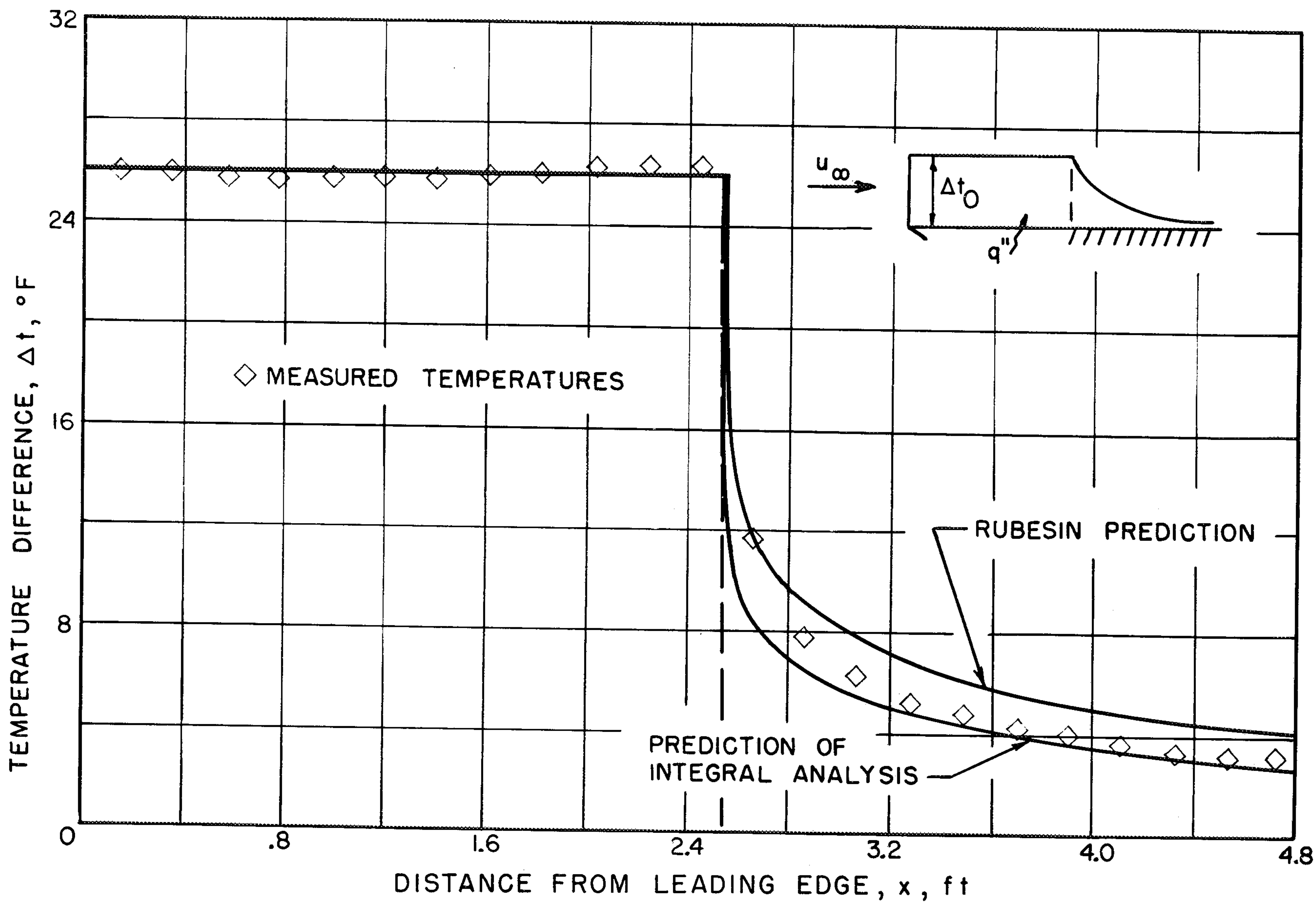

Figure 9. - Example for constant temperature followed by adiabatic wall.

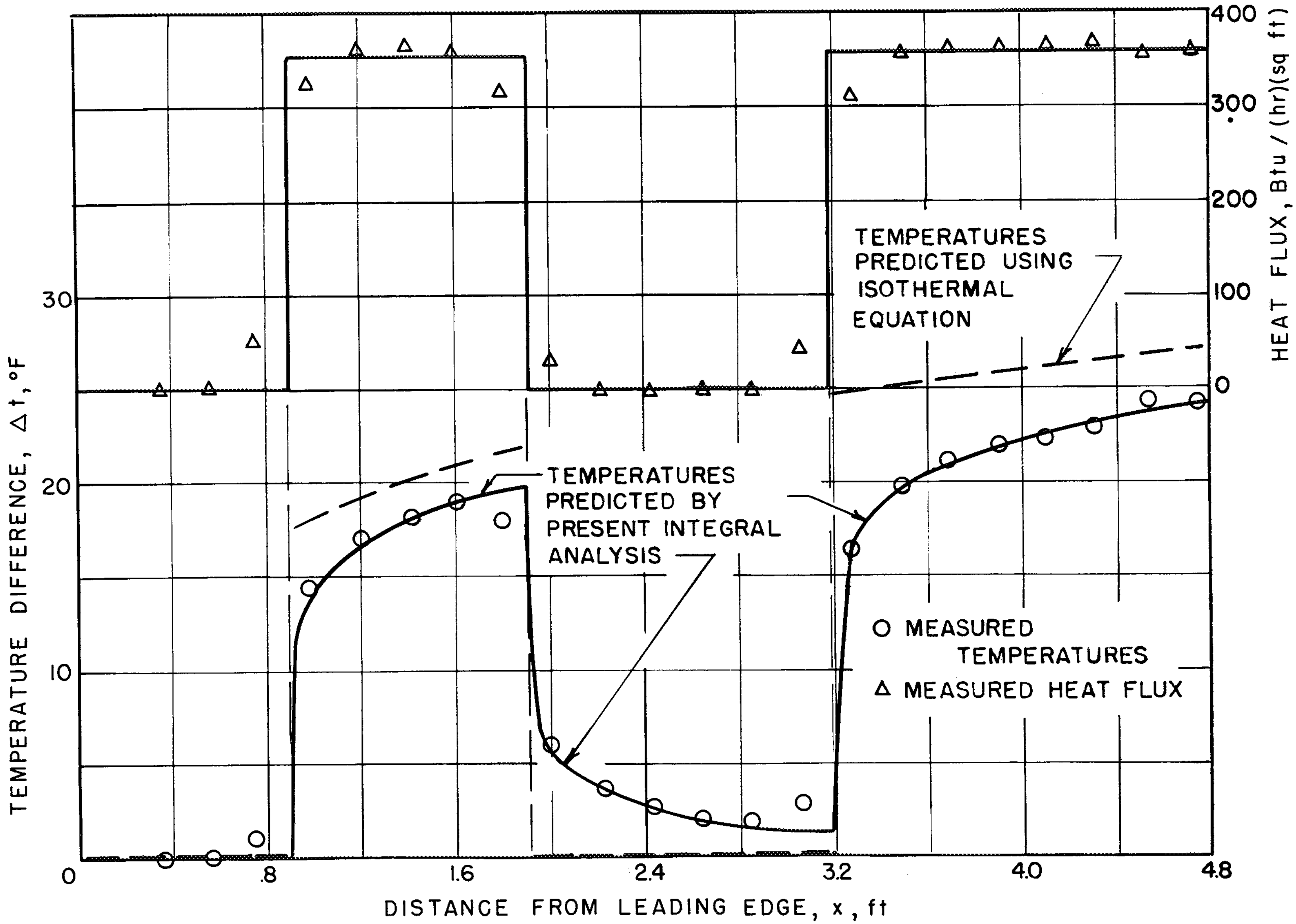

Figure 10. - Example for pulse heat input.

4994

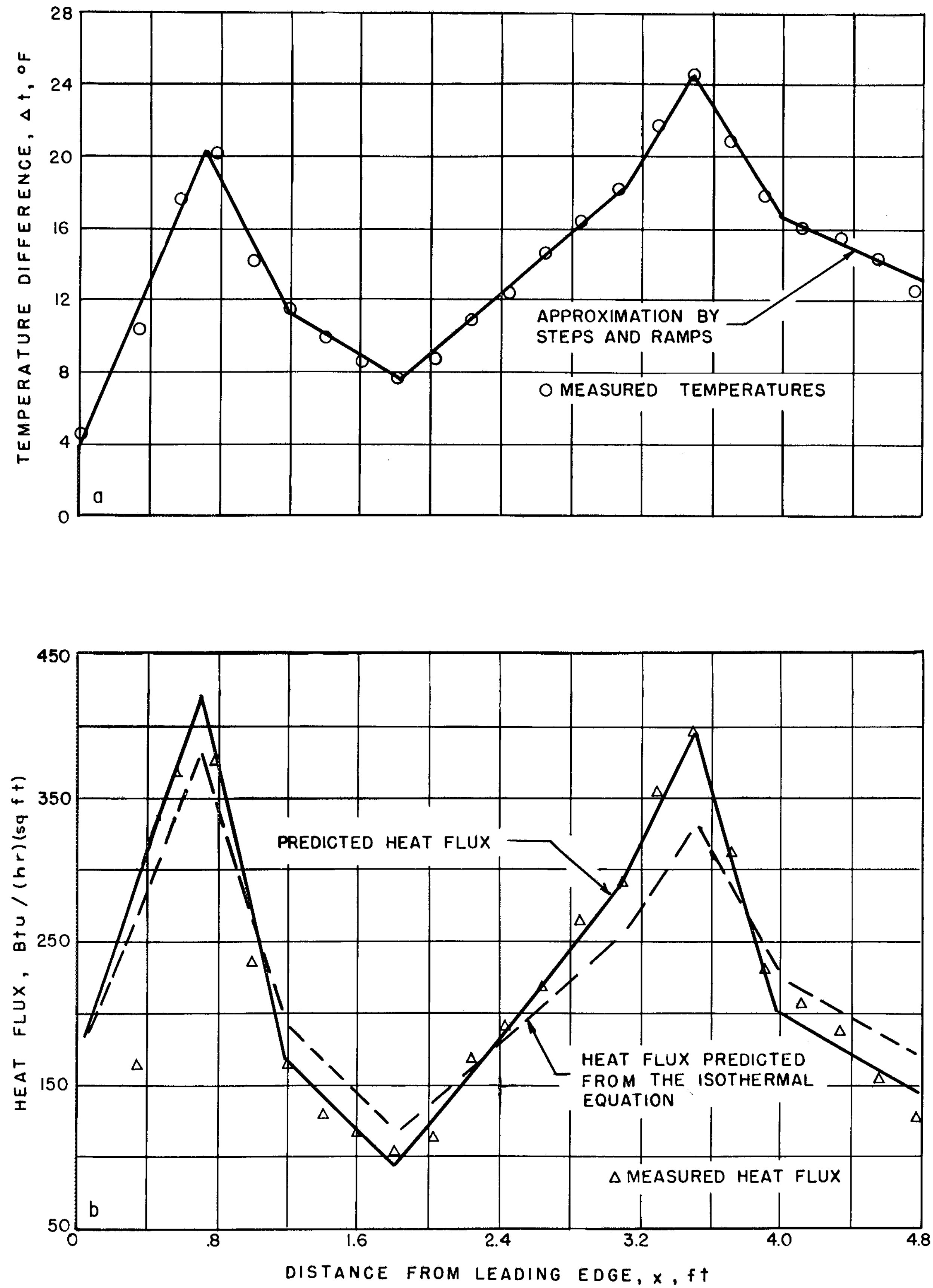

Figure 11. - Prescribed temperature treatment of irregular example.

4994

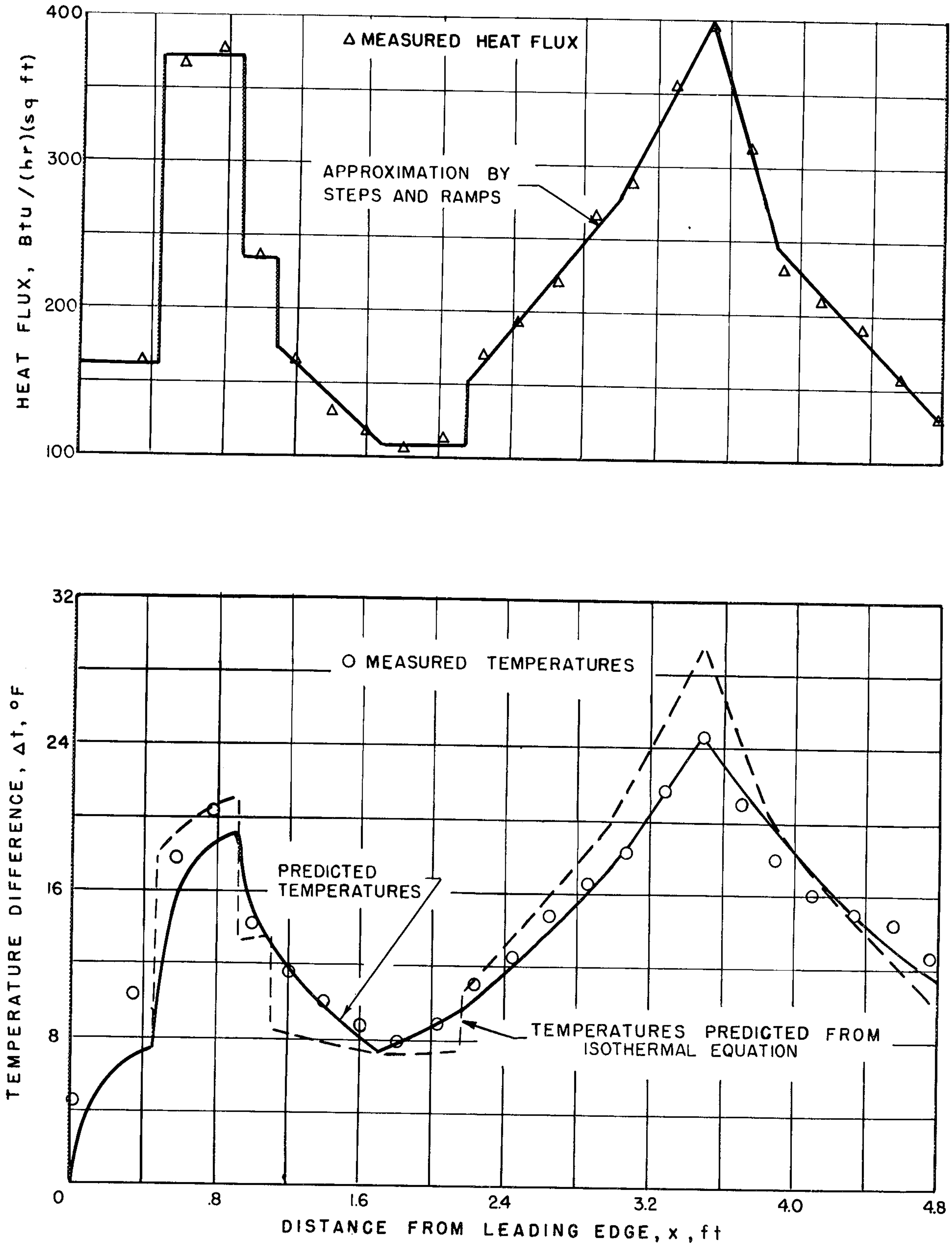

Figure 12. - Prescribed heat-flux treatment of irregular example.